Developmental Mathematics Students' Use of Representations to Describe the Intercepts of Linear Functions

Anne Cawley
University of Michigan

This paper reports findings from a pilot study that investigated the way that six college students enrolled in a developmental workshop worked through a task of nine problems on linear functions. Specifically, I investigated two aspects, the order that students completed the problems and what sources of information the students used to find the requested features, and also the types of representations (symbolic, graphical, or numerical) students used to describe the intercepts of the function. Findings suggest that students have an overwhelming reliance on the graph of the linear function and that there is variation in the number of representations used to describe the intercepts (Single, Transitional, and Multi Users). Because the graphical representation is a preferred representation, it may be wise to build student knowledge from this representation, making connections to other representations. This study contributes to understanding the mathematical knowledge that developmental mathematics students bring to the classroom.

Key words: Representations, Linear Functions, Developmental Mathematics

It is well known that the concept of function is central to mathematics. Functions are not only a central part of secondary mathematics curriculum, but also a concept that continues to grow in importance as students continue their mathematical studies in postsecondary education (Yerushalmy & Schwartz, 1993, p. 41). Students entering into developmental mathematics courses have been exposed to the concept of linear functions throughout their academic careers. In order to expand the conceptions that students bring with them, it is valuable to first know what these conceptions might be. While there is much research on developmental mathematics, Stigler and Givvin (2009) state that we as a field know almost nothing about what developmental mathematics students actually know. As a consequence, we cannot change instruction to meet the needs of our students if we do not really understand what their needs are. If we cannot change our instruction to support the needs of our students, then there will be no change in the current status of developmental mathematics courses.

As a way to address the knowledge that students bring into a developmental mathematics course, I designed this study to focus on describing how students work on problems about linear functions. In particular I was curious to know how students work through a set of problems and what types of representations they tend to use to do so. Because various representations can be used to find features of linear functions, I specifically analyze the types of representations students use when describing the intercepts of a linear function. I discuss how students engage with various features of linear functions, the types of representations that are used when describing intercepts, and how this helps to identify the conceptions that students may have of linear functions upon entering into their first developmental mathematics course. First I discuss the literature on developmental mathematics, representations in mathematics, and linear functions. Next I describe the methods and findings for the participants, followed by the discussion and conclusion.

Literature Review

Developmental (or remedial) mathematics is an important area of postsecondary mathematics education. Mathematics is the most common subject requiring developmental coursework. Developmental courses are often viewed as a gatekeeper for students to make progress to degree completion (Attewell, Lavin, Domina, & Levey, 2006; Bettinger & Long, 2005), but are also a pre-requisite for many students aiming to enter technical fields (e.g., STEM, business). Remedial coursework predominantly addresses the content that students are lacking instead of acknowledging what students already know, and can use, to build upon their knowledge (Coben, Fitsimons, & O'Donoghue, 2000). This deficit view is starting to be challenged; developmental mathematics courses currently focus on re-teaching students content that they do not "know" (Sitomer, 2014). However, more research needs to be done to understand what knowledge students *do* bring with them as they take developmental mathematics courses such that efforts can be made to consider how such to incorporate prior knowledge into instruction.

One way to understand student knowledge is through the representations that students use to convey mathematical ideas. Representations can be thought of as tools for thinking, explaining, or justifying a student's understanding (Pape & Tchoshanov, 2001) and as such, they may provide insight for researchers and teachers in regard to what students understand. There are four generally accepted representations in mathematics: verbal, numerical, graphical and symbolic (Friedlander & Tabach, 2001). The *verbal* representation is normally used when posing a problem or may be needed as a final interpretation of the result of a mathematical problem. The *numerical* representation can be seen as a predecessor to more generalized representations in mathematics. Generally, numerical representation is used to gain a strong understanding of a problem and as a way to investigate particular cases. The *graphical* representation provides a visual picture of mathematical phenomena, such as a function. The *symbolic* representation is the algebraic form of patterns or mathematical models, generally in the form of an equation.

The use of representation in the learning of mathematics is important. Coulombe and Berenson (2001) argue that in many ways, representation is the language of mathematics and that "fluency with multiple representations of mathematical relationships plays a significant role in the successful development of algebraic thinking" (p. 172). However, facility to translate across multiple representations is not always natural. The ability to interpret and translate among multiple representations is a tool that can aid students in demonstrating the mental constructs they may hold of deep mathematical concepts, such as functions.

Knuth (2000) gave 178 high school students a set of problems where they were given both a symbolic and graphical representation of a function. In most problems, the graph was the more efficient representation to use when solving the task. He found that students were more prone to use the equation of the line to provide their answers. He suggested that students may feel that the graph hides the necessary information, and by doing so, do not see the function from the process perspective (made up of individual points) and rather see it from the object perspective (see it as one line, not made up of points). Knuth claims that this may be due to students' belief that the symbolic representation provides more accurate solutions whereas when using a graphical representation, students may feel they need to estimate a response and therefore compromise accuracy. Because of this, students do not see the necessary connections between a graph and its corresponding symbolic representation and have a tendency to consider functions in the equation-to-graph direction. In conclusion, Knuth argues that

> an important aspect of developing a robust understanding of the notions of function means not only knowing which representation is most appropriate for use in different contexts but also being able to move flexibly between different representations in different translation directions (p. 53).

It is clear that an important way to better reveal a student's understanding of a concept is to recognize the types of representations he or she uses to describe or explain their understanding. Being able to use multiple representations provides for more fluent mathematical comprehension as well as provides flexibility when working with a mathematical idea. Linear functions, in particular, provide many opportunities for flexible use among verbal, numerical, graphical and symbolic representations, and therefore can demonstrate the type of understanding that a student holds. In order to gain a better understanding of what knowledge of linear functions students bring to a developmental mathematics course I investigated the following research questions: 1) What are the features of linear functions that developmental mathematics students draw on to answer questions on linear functions? and 2) What types of representations do students use when describing the x- and y-intercepts of a linear function? For purposes of this paper, I focus my discussion on Research Question 2.

Theoretical Framework

I ascribe to Vygotsky's (1978) sociocultural theory of learning, which makes two key assumptions: 1) learning is socially embedded and 2) what a student can do with the assistance of others can make clear the knowledge that a student has of mathematical content. It is a naïve assumption that students enter the classroom as *tabula rasas;* all students enter the classroom with preconceptions of mathematical content (Confrey, 1990). At times, this knowledge may not be captured by a placement test. I believe that what a student knows can be better exposed by interactions with someone who has a control of the content.

Because learning is socially embedded, Vygotsky (1978) introduced a different type of developmental level, the *zone of proximal development* (ZPD). Vygotsky believed that what a student can do with the assistance of their teacher or peer could be more indicative of their capabilities than what he or she can do in isolation. Vygotsky argued that this zone defines the knowledge that has not quite yet matured, but is in the process of maturation indicating that the student will achieve such knowledge in the near future.

Developmental mathematics students are deemed to lack such maturity of mathematical knowledge yet they are expected to be able to demonstrate a control of such knowledge by the end of a course. It is for this reason that I choose to frame this study using Vygotsky's sociocultural theory of learning, and in particular to create a space (or "zone") where this type of knowledge can become more apparent.

Methods

The study took place at a large, public, four-year institution on the West coast of the United States. In 2013, this university admitted a total of 19,682 first-time freshmen. Per university requirement, all admitted students mandated to enroll in a developmental mathematics course must enroll in a summer workshop prior to matriculation in the fall term. The summer workshop is a non-credit bearing course equivalent to a full semester of a developmental mathematics course. Students receive credit for the workshop by earning an equivalent grade of a "C" or better; they receive no credit if they earn an equivalent grade of a "D" or lower.

The participants in this study were recruited from the summer workshops. A total of 687 students were required to enroll in a workshop in summer of 2013. I recruited participants from two sections of MATH 20 taught by the same instructor. The course met for a total of four weeks. The class was three hours a day and met five days a week. The students recruited for the study were recent high school graduates. Six students agreed to participate in the study—Carina, Catherine, Ramona, Veronica, Carter, and Nick.

Procedure

I designed a task that contained nine problems on linear functions. Each problem required students to provide missing information about nine different linear functions. I presented problems in the form of a table, with column headers listing the features that were manipulated: the graph, the equation of the line, *y*-intercept, *x*-intercept, a point on the line, a point not on the line, if the line was increasing or decreasing, and the slope (see *Figure 1*). Each row represented a different linear function. I provided some information in the cells and the students' objective was to fill in the missing information for each row. For example, the first row, which represents Problem 1, asks students to find the graph, equation of the line, *x*-intercept, a point on the line, a point not on the line, and whether the line is increasing or decreasing given a *y*-intercept of (0, 0) and a slope of 1.

Please fill out each row of the following table given the information provided.

Problem #	Graph	Equation of the Line	*y*-intercept	*x*-intercept	A point on the line	A point *not* on the line	Increasing or Decreasing	Slope
1			(0, 0)					1
2				(0, 0)	(1, -3)	(2, 6)		
3		3x = 2y						
4	Graph of y = -2x – 1							
5			(0, -1)		(3, -1)			
6		4x - 2y = -1						
7		y - 3 = - 2(x - 1)						
8		x - 2 = 3(y - 1)						
9				(-2, 0)	(-2, -3)			

Figure 1: Problems used in session. Note: The cells in the graph column contained a blank coordinate plane. The graph cell in Problem 4 contained a coordinate plane with the line $y = -2x - 1$ *graphed.*

The students worked individually on the task and were asked to "think aloud" while working through the task. The think aloud format allowed me to better understand how they engaged with the problems in the task. I listened actively and asked questions to clarify meaning. When the participants were quiet, I urged them to express out loud what they were thinking or asked an open-ended question about their work (e.g., "How did you know the slope was 3?" "What are you thinking right now?"). When a student claimed that he or she did not know how to provide a response within a specific problem, I asked questions to probe what they were able to recall about that particular feature (e.g., "What do you know about an equation of the line?" "You said you do not remember what an *x*-intercept is. What can you tell me about a *y*-intercept?"). Because I ascribe to a sociocultural theory of learning, I believe that the questions I asked the students during the problem solving session were important and necessary so as to better demonstrate the knowledge that students may have had of linear functions. Without such questions, I may not have captured such knowledge. Each session was scheduled to last 30 minutes. The interviews were audio and video recorded. The transcripts of the video recordings included descriptions of the participants' actions. Because I am interested in the knowledge that students bring with them prior to formal instruction I had planned to conduct the problem solving sessions during the week prior to the start of the workshop prior to any exposure to linear functions.

Analysis

To answer the Research Question 2, I analyzed the data in two ways. In the first analysis, Intercepts by Student, I recorded the types of representations that individual students used to find the intercepts in all problems they attempted. In the second analysis, Intercepts by

Representation, I looked at all of the instances where a specific representation was used to describe all of the ways that students used the representation. I created a coding scheme based on Friedlander and Tabach's (2001) labeling of representations to categorize students' justifications for the responses given for the *x*- or *y*-intercept. The scheme contained three codes: 1) graphical, 2) symbolic and 3) numerical. I did not use Friedlander and Tabach's (2001) verbal code because the think-aloud nature of the session required all students to provide a verbal justification.

I coded each student's description or justification of a response given for the *x*- or *y*-intercept across all problems attempted. Specifically, I coded the justification by how they represented the intercept in these descriptions—graphically, symbolically, or numerically. I coded a justification as graphical if a student used the graph as a part of their reasoning (e.g., a student could describe the intercept as a point on the graph or may run their finger up or down an axis to indicate a location). I coded a justification as symbolic if a student used the equation of the line as a part of their reasoning (e.g., a student could indicate knowing that the b in the form $y = mx + b$ is the *y*-intercept or may know that to find an intercept, you must plug zero into the opposite variable and solve). I coded a justification as numerical if a student provided a value for the intercept without reference to either the graph or the equation of the line (e.g., a student might just give a value and just say they know it to be true or if given that one intercept is 0 the other intercept must be the same). If a student could not provide a response, I did not assign a code. A student's justification could be marked with more than one code.

I organized the results first by student to look for similarities in the ways students use the representations to describe the intercepts. I then look across each representation type individually (symbolic, graphical, numerical) to describe the ways that each representation was used. I checked the reliability of the coding scheme by having two other researchers use apply it resulting with 100% agreement with each researcher.

Findings

Table 1 shows the overall frequencies of the types of representations a student used to describe the intercepts (XI for *x*-intercept and YI for *y*-intercept). G, S, and N indicate whether a student used a Graphical, Symbolic, or Numerical representation to justify their response. Bold and italicized letters in the table indicate that the student did not provide a correct response. As can be seen in *Table 1*, Carina and Catherine primarily used one type of representation. Carter and Nick used both symbolic and graphical representations, however, gave numerous descriptions for what an intercept is throughout the task. Ramona and Veronica used both symbolic and graphical representations, providing the same description of the intercept throughout the task.

Now I look at three representations (symbolic, graphical, and numerical) to describe the ways in which each of the representations were used across all students. There were two ways in which symbolic representations were used: first, students use the equation to substitute values and find other values; second, students used the parameters in the equation to give values. In addition, students used a symbolic representation more often when finding the *y*-intercept. In contrast, when finding the *x*-intercept, students tended to use a graphical representation, using a symbolic representation less. This is important, as it appears that students rely on the parameter b from the slope-intercept form when finding the *y*-intercept but avoid the symbolic form as often to find the *x*-intercept.

Students also used both methods incorrectly to find the intercepts. For example, Catherine used the symbolic representation to find the *x*-intercept by claiming that the value m indicated the *x*-intercept. Carter also said that in order to find an intercept, you could substitute any value for the opposite variable and solve, but that 0 was the easiest number to use. Carina did not use symbolic representations to find the intercepts (she used only graphical representations) and Catherine used only symbolic representation to find the intercepts. This shows a reliance on one type of representation to find the intercepts, which may lead to incorrect responses.

Table 1

Frequencies of Representations Used

Problem	Carina		Catherine		Ramona		Veronica		Carter		Nick	
	XI	YI	XI	YI	XI	YI	XI	YI	XI	YI	XI	YI
1	G		***S***[b]		G		G		G, S, N		G	
2		G		***S***		G		G		G, S		G, S
3	--[a]	--	***S***	S	G	S	G, S	S	--	--	G	G, S
4	G	G	***S***	***S***	G	G	G, S	S	G, S	G, S	G	G
5	G		***S***		G		G				G	
6					S	S						
7					G	S						
8					G, S	S						
9					S							

Note: [a] Indicates that a student did not provide an answer nor indicated the needed information to find the intercept. [b] Bold, italicized letters indicate that the student gave an incorrect response. Light gray cells indicate that the intercept was given information in the problem. Dark gray cells indicate that the student did not attempt the problem. The table is arranged by gender and then alphabetically within each gender category.

There was one common way that students used a graphical representation: they indicated they were looking for a specific location, a point, on the graph were it crossed the *x*- or *y*-axis. Students would find this point in one of two ways. The first way was by running their pencil along the *x*- or *y-axis* to locate the point that lay on the axis. The second was by run their pencil along the graph of the *line* they were looking for to find where it touched the axes. The graphical representation was the most widely used representation when finding the intercepts (see *Table 1*) and was used more often when finding the *x*-intercept.

Students also used a graphical representation incorrectly to find the intercept. Carter stated that an intercept was a vertical or horizontal line, not a point, whereas Nick stated that any point plotted on the grid lines on the Cartesian plane could be the *x*-intercept. Both students, however, adjusted these descriptions of the intercepts in Problem 4, which gave the graph. Thus perhaps when required to provide a graph on their own, a student may struggle to describe his or her ideas of the intercepts.

There was one way in which the numerical representation was used, to indicate that if the *y*-intercept was (0, 0), then the *x*-intercept must also be (0, 0). Only one student used the numerical representation in this way. Students did not use numerical representations to find the intercepts in general. The first three problems included lines with *x*- and *y*-intercepts of (0, 0). I had assumed that students might provide a number for an intercept if they could not use a symbolic or graphical representation to justify a response, however, no student did this.

Discussion

Upon analysis of the data, I have categorized the students into three groups according to their use of representations when identifying the intercepts. The first group, the Single Users, includes Carina and Catherine, who relied on one representation only when justifying their responses for the intercepts. The second group, the Transitional Users, includes Carter and Nick, who used multiple types of representations when describing the intercepts, and who in addition, changed their definition of an intercept as they worked through the task. The third group, the Multi Users, includes Ramona and Veronica, who used a mix of graphical and symbolic representations to justify their responses for the intercepts.

The Single Users depended on one representation when describing the intercepts. This reliance on a single representation indicates to me that these students do not have a flexible (Arcavi, 2003) or fluent (Coulombe & Berenson, 2001) understanding of the concept of intercept. A flexible understanding of the concept of intercept is manifested by the use of multiple representations, as needed, to describe it. Catherine appeared to only see the intercepts as coefficients found in the equation of the line written in slope-intercept form and does not indicate that the intercepts are also features of the graph.

The Transitional Users used multiple types of representations when describing the intercepts *and* their definition of the intercept shifted throughout the task. In this group, both students provided inaccurate definitions of intercepts until they reached Problem 4, which provided the graph of a function. Upon working through Problem 4, both students realized that their previous work was not correct. Their concept of intercepts transitioned, which is similar to Moschkovich's (1998) idea of a transitional concept. Carter and Nick started with one definition of an x-intercept (for Carter, they were vertical lines and for Nick, they were the points that fell on the gridlines) and through their work on Problem 4 (where the graph was given) revised their definition. This indicates that their understanding of the intercept may be unstable with problems that do not have a graph of a line and more stable with problems that have the graph of the line.

The Multi Users used both graphical and symbolic representations to justify their responses for the intercepts. Ramona and Veronica used the same definitions of intercepts throughout the problem set, using two types of representations to provide responses. These students tended to use one representation at a time to justify finding an intercept. However, in a few instances, they used two representations to describe an x-intercept, and did so to verify that the response was correct. These students were aware of the given information in the problems and capitalized on this information to justify the responses that they gave by using different representations in different problems. While these students did not use a numerical representation, they still used more than one representation when referring to the concept of intercept, which indicates that this group may have a more flexible understanding of the concept than the Single Users.

This typology of users characterizes different ways that students rely on representations to justify their responses for intercepts, which may reveal their understanding of the concept. Single User students may have insufficient knowledge of intercepts and therefore use the representation they know as justification for their solutions. However, relying on a single representation is not an effective way to consider the intercepts of linear functions and does not imply flexibility. Transitional Users may exhibit an uncertain understanding of the definition of an intercept when required to provide the graph of a linear function and may show a reliance on being given the graph to assist them in recollection of the concept. Multi Users may have a better grasp of the concept of intercepts because they can demonstrate a flexibility of use of representations as needed within the problem sets.

As Stigler and Givvin (2009) pointed out, there is not much research that focuses on what college students *do* know when they enter developmental mathematics. This study aimed to begin conversations about what college students taking developmental mathematics know about linear functions. The findings from this study indicate that students use varying amounts of representations, and show a preference for graphical representations. This implies that it may be wise to build student knowledge from the graphical representation in developmental mathematics courses, therefore making stronger connections to other representations.

References

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational studies in mathematics*, *52*(3), 215-241.

Attewell, P., Lavin, D., Dominia, T., & Levey, T. (2006) New evidence on college remediation. *The Journal of Higher Education*, 77(5), 886-924.

Bettinger, E. P., & Long, B. T. (2005). Remediation at the community college: Student participation and outcomes. *New Directions for Community Colleges*, *2005*(129), 17-26.

Coben, D., FitzSimons, G. E., & O'Donoghue, J. (2000). *Perspectives on Adults Learning Mathematics: Research and Practice*. Hingham, MA: Kluwer Academic Publishers.

Confrey, J. (1990). A review of the research on student conceptions in mathematics, science, and programming. *Review of research in education*, 3-56.

Coulombe, W., & Berenson, S. (2001). Representations of patterns and functions: Tools for learning. In Cuoco, A. A., & Curcio, F. R. (Eds.). The roles of representation in school mathematics. National Council of Teachers of Mathematics.

Friedlander, A., & Tabach, M. (2001). Promoting multiple representations in algebra. In Cuoco, A. A., & Curcio, F. R. (Eds.). The roles of representation in school mathematics. National Council of Teachers of Mathematics.

Knuth, E. (2000). Understanding connections between equations and graphs. *The mathematics teacher*, 93(1), 48-53.

Moschkovich, J. N. (1998). Students' use of the x-intercept as an instance of a transitional conception. *Educational Studies in Mathematics*, *37*(2), 169-197.

Pape, S. J., & Tchoshanov, M. A. (2001). The role of representation (s) in developing mathematical understanding. *Theory into practice*, *40*(2), 118-127.

Sitomer, A. (2014). *Adult returning students and proportional reasoning: Rich experience and emerging mathematical proficiency.* Unpublished Doctoral dissertation, Portland State University, Portland, OR.

Stigler, J. W., Givvin, K. B., & Thompson, B. J. (2010). What community college developmental mathematics students understand about mathematics. *MathAMATYC Educator*, *1*(3), 4-16.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes* Cambridge, Mass.: Harvard University Press.

Yerushalmy, M., & Schwartz, J. (1993). Seizing the opportunity to make algebra mathematically and pedagogically interesting. In Romberg, T.A., Fennema, E., & Carpenter, T.P., (Eds), *Integrating research on the graphical representation of functions*. (pp. 41 – 68). Hillsdale, NJ: Erlbaum.