An interconnected framework for characterizing symbol sense

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Algebraic notation can be a powerful mathematical tool, but not all seem to develop "symbol sense," the ability to use that tool effectively across situations. Analysis of interview data with both novice and expert users of notation identified three interconnected viewpoints: looking at, with, and through the notation. The framework has implications for instruction and potential development of symbol sense.

Key Words: symbol sense, algebraic notation

Algebraic notation can be a powerful mathematical tool, yet not all students seem to develop "symbol sense." The ever-changing state of technology contributes to the motivation for mathematics educators to define symbol sense and design instruction to encourage its development. In particular, the *Common Core State Standards for Mathematics* (2010) calls for students to be able to both *decontextualize*—work with abstract symbols while allowing the referents to shift to the background—and *contextualize*—to reconnect with those referents as needed in order to appropriately interpret the relationships within the situation.

Background and Methods

Mathematical symbols, and algebraic notation in particular, can be the *focus of* one's attention, or the *means through which* one's attention on quantitative relationships is mediated. The ability to make and attend to such shifts can be considered *symbol sense*, or a coherent approach to algebraic notation that supports and extends mathematical reasoning. Symbol sense thus goes beyond efficient manipulation of symbols to being able to select, construct, manipulate, and interpret notational forms in service of mathematical work (Author). While the importance of developing such symbol sense is widely accepted, the process by which this happens is not yet well understood. Arcavi (2005) identifies three open issues related to the development of symbol sense. First, we do not have a full characterization of symbol sense, in terms of having a comprehensive set of categories to inform research and instruction. The second issue may be cast as *nature versus nurture*: can symbol sense be taught and learned, or are there "symbol experts" that have an inherent sense of symbols? The final issue addresses the interplay between technical practice and symbolic reasoning; that is, how does technical fluency interact with the development of symbol sense? These three issues are not trivial and will not be answered easily. Recent studies (cf. Banarjee & Subramaniam, 2012; Hewitt, 2012; Bokhove & Drijvers, 2012) have explored options for designing algebra instruction; reasoning about structural aspects of the notation is a common theme and resonates with the idea of developing symbol sense.

Views from previous work (Kinzel, 2000) were combined with Arcavi's characteristics of symbol sense (1994, 2005) in the design of the current study. An interconnected framework emerged that coordinates three viewpoints: *looking at*, *with,* or *through* the notation. *Looking at* the notation can involve noticing particular aspects and considering appropriate actions; more fluent users are able to consider a wider range of possible actions. An ability to *look with* the notation connects with Arcavi's notion of "friendliness" with symbols, that the individual recognizes the power of a symbolic representation and that symbols are readily available as a means of representation. Coordination of *looking at* and *with* the notation may involve conscious choices; for example, having chosen one representation, the individual is capable of considering the implications—either for representation or for manipulation—and perhaps changing or altering the initial choice. Arcavi also includes the ability to "read through" the symbols to assess affordances of manipulations, or to check for meanings within the implementation of procedures. Interview data supported the characterization of a third view, *looking through* the notation to explore underlying relationships. Effective work with symbolic forms on a single task can involve conscious coordination of these viewpoints.

Task-based interviews were conducted with 11 students enrolled in a proof-based mathematics course and with 9 practicing mathematicians. Data were analyzed in terms of the introduction of symbols, construction of expressions and/or equations, manipulation of symbolic forms, and interpretation of intermediate or final results, as well as any shifts in focus of attention or use of notation (e.g., choosing to change the nature of a representation). Explicit articulations related to interpretation or use of notation were noted and categorized. Narratives were created for each interview, capturing the individual's use of and articulations about the notation. The analysis of these narratives lead to the proposed framework, which characterizes three viewpoints (*looking at, with,* and *through* the notation) and interactions between them.

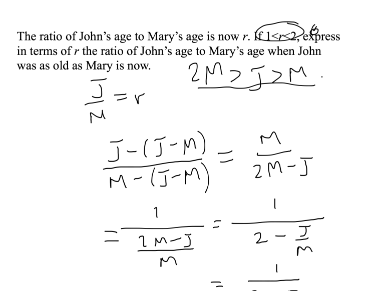
Sample Task Selection and Analysis

Tasks were carefully selected for the interviews, in order to be accessible to a range of participants but to also provide enough complexity so that a participant's approach to notation becomes apparent and an explicit focus of the interview. The Age Ratio task was used for all participants; a brief analysis of the task provides an illustration of overall task selection: *The ratio of John's age to Mary's age is now r. If 1<r<2, express in terms of r the ratio of John's age to Mary's age when John was as old as Mary is now.*

This task presents a concise yet complex set of information. The relationship between the ages now and at a point in the past are defined, but in an abstract manner rather than through specific numeric values. The relevance of the given restriction on the value of *r* is not necessarily immediately apparent. A successful response to the task involves constructing expressions for the ages *then* in terms of the ages *now* so that the new ratio can be expressed in terms of the current ratio. This requires noticing (1) that the point in the past is when John's age was equal to Mary's current age and (2) that the number of years between *now* and *then* is equal to the difference in their current ages. Once an appropriate expression for the ratio is constructed, the choice of manipulations to express this in terms of *r* is not necessarily immediately apparent. Thus, the participant may need to choose between options, and their thinking about such choices can be explored. The resulting expression for the ratio of ages *then* $\left(\frac{1}{2-r}\right)$ can be interpreted in terms of the relationship between the two ratios; the relevance of the restriction on *r* may also now be apparent. A potential difficulty with this task is the assumption that the ratio between ages will remain constant; such an assumption makes expressing the ratio of ages *then* nonsensical.

A mathematician's work

The following data excerpt illustrates the framework through the work of a fluent symbol user on the Age Ratio Task (see Figure 1). M6, a



practicing mathematician, read the task statement and wrote $\frac{J}{M}=r$ as he read. He interpreted the interval $1<r<2$ to mean that John is older than Mary and rewrote this as $2M<J<M$. Without much articulation, he then determined that the difference in their ages is a relevant quantity and can be represented as *J-M*. The ratio of ages "then" is expressed as $\frac{J-(J-M)}{M-(J-M)}$ and simplified to $\frac{M}{2M-J}$; as he simplified, he was pleased that (1) the denominator will be positive within the given restriction on *J* and *M* and (2) John's age then (the numerator) simplified to Mary's age now (*M*). At this point, he evaluated his work in relation to the goal of expressing this ratio in terms of *r*: "I need to manipulate that to be in terms of r, or solve for one of them in terms of the other." He chose to divide through by *M* and obtained $\frac{1}{2-r}$. He was pleased with this result "especially as it agrees nicely" with the given interval for *r*. "Barring typos," he is confident that he has an appropriate solution to the task.

Applying the framework

In terms of the viewpoints, M6 introduced the symbols *J* and *M* to *look with* and record given information. In interpreting the interval in terms of the ages, he demonstrates the ability to *look through* the notation to see a relationship. Determining that the desired ratio can be expressed by subtracting the difference in ages (*J-M*) from each age allows him to once again *look with* notation. Part of this *looking with* includes noting alignment with the context. His statement about needing to manipulate this ratio indicates a shift to *looking at* the notation and making choices related to the goal. Noting that his final expression "agrees nicely" with the interval indicates a tendency to continue to *look with* the notation and check meaning within the task context. It would be possible to consider the relationship between this ratio and the given ratio (*r*), which would be an instance of *looking through* the notation. M6 did not do this, but it was not explicitly required by the task.

In contrast, a less fluent user may be distracted from one viewpoint by another. For example, one student participant (S1) was asked to solve this system of equations for *x* and *y*: $xy=100$ and $(x-5)(y-1)=100$. Within his work, he produced the linear relationship: $x-5y=5$. This was unexpected (he had anticipated being able to solve for either *x* or *y* directly) and prompted him to find the *x*- and *y*-intercepts for this line. When asked if he had solved the system, he expressed surprise that these intercepts are not solutions to the original equations. In this case, the participant's fluency with a known procedure seemed to interfere with his ability to interpret the notation within the context, although he did attempt to *look through* the notation. The intercepts are not solutions to the system, but he did believe that the solutions will lie somewhere on this line. However, he incorrectly connected this to the first equation, stating "I keep coming back to [$xy=100$]. Somewhere along that line, the solution's going to be 100. But I don't know how to find it." When asked how he knows this, he states: "Because I was able to come up with an equation [referring to $x-5y=5$]. There you go." This comment indicates that his observations from *looking at* the notation take precedence over any attempts at *looking through*. Such instances seem related to Arcavi's third issue, the interaction between technical fluency and appropriate interpretation; the triggered known procedure was applied in spite of not having a clear connection to the context.

Implications

Constructing narratives of individuals' work served to refine the viewpoints within the framework. Including both novice (undergraduate students) and expert (mathematician) participants expanded the range of actions described by the framework. Linking the views through the framework emphasizes seeing notation as a tool to support mathematical reasoning. This may sit in contrast to instructional approaches in which manipulations are the primary focus of attention. Time spent developing fluency with specific procedures can strengthen one's ability to recognize and evaluate the potential of particular forms. Without the complementary views of *looking with* and *through* the notation, however, these manipulations can be empty processes. The framework can inform instructional design, in that the views can be incorporated into task selection. As with the interviews in this study, tasks can be evaluated in terms of their potential to provide opportunity for or even require explicit attention to shifts between and coordination of views, thus potentially contributing to the development of symbol sense.

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