Calculus students' deductive reasoning and strategies when working with abstract propositions and calculus theorems

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In undergraduate mathematics, deductive reasoning is an important skill for learning theoretical ideas and is primarily characterized by the concept of logical implication. This plays roles whenever theorems are applied, i.e., one must first check if a theorem's hypotheses are satisfied and then make correct inferences. In calculus, students must learn how to apply theorems. However, most undergraduates have not received instruction in propositional logic. How do these students comprehend the abstract notion of logical implication and how do they reason conditionally with calculus theorems? Results from our study indicated that students struggled with notions of logical implication in abstract contexts, but performed better when working in calculus contexts. Strategies students used (successfully and unsuccessfully) were characterized. Findings indicate that some students use "example generating" strategies to successfully determine the validity of calculus implications. Background on current literature, results of our study, further avenues of inquiry, and instructional implications are discussed.

Key words: Logic, Implication, Calculus, Theorems, Conditionals

Background and Research Question

Calculus plays a fundamental role in many science, technology, engineering, and mathematics (STEM) areas such as physics and engineering. Thus, many STEM majors will take at least one semester of calculus as part of their major, during which they will encounter propositions, lemmas, and theorems. For example, students encounter the "If a function is differentiable at a point, then it is continuous at that point" theorem. Students must then apply this theorem in a variety of situations, such as when they are given a function that is differentiable or when they are given a function that is continuous. This deductive process, characterized by logical implication, is a hallmark of mathematical thinking. It seems natural to assume that to use a theorem effectively, a student must comprehend logical implication, which requires the understanding of the four classic reasoning patterns. These patterns are provided below with the assumption that the rule "A implies B" holds.

Modus ponens: Suppose A is True. Then B is True.
Inverse: Suppose A is False. Then it is not known whether B is True or False.
Contrapositive: Suppose B is False. Then A is False.
Converse: Suppose B is True. Then it is not known whether A is True or False.

Applying this reasoning can enable a student to know, for example, that a function being continuous at a point does not necessarily imply that it is differentiable at that point.

It is well-established that both children and adults struggle with these kinds of logical reasoning tasks (O'Brien, Shapiro, & Reali, 1971; Wason, 1968). However, it appears that people are more successful when the questions are posed in a context (as opposed to abstractly) (Stylianides, Stylianides, Philippou, 2004). Also, it is well known that students struggle with calculus ideas such as limits, differentiation, and integration (e.g., Carlson & Rasmussen, 2008; Tall, 1993; Orton, 1983; Zandieh, 2000). The instruction students receive about these key calculus ideas often includes theorem or theorem-like statements and students are expected to reason logically from them. Although much work has been done separately on the issues of logical implication and calculus learning, we know little about how students engage with logic tasks that are set in a calculus context. In particular, we were interested in whether calculus students had the same kinds of difficulties with calculus-based tasks as they did with the purely abstract tasks. In other words, are calculus theorems enough of a "context" to support students' productive reasoning or are those tasks treated in the same way as the classical, abstract tasks? This research project was designed to examine the following questions: How successful are calculus students with logical implication tasks set in calculus and abstract contexts? What strategies do students use when engaged in calculus theorem tasks involving logical implications? Answers to these questions can provide insights into student sense-making that can be then used to inform instructional design aimed at improving student understanding of theorems and definitions in calculus.

Research Methods

Similar to much of the prior work on student thinking about calculus, this study was done from a cognitive theoretical perspective and thus students' written and spoken statements were used as data on their thinking and understanding of the ideas. Surveys were given in a first semester differential Calculus I class at a university in New England near the end of the fall semester. There were a total of 52 participants. The surveys consisted of two parts. Part I consisted of calculus theorem tasks that were modeled after the four reasoning patterns on the previous page. In Part II, the same four tasks were given but presented in an abstract manner. Many of these tasks resembled syllogisms (e.g., All men are mortal. Socrates is a man. Therefore, Socrates is mortal) but were stated in a formal context using letters and symbols to represent statements. See Figure 1 for sample tasks. Although other researchers have established the difficulties students have with these kinds of abstract tasks, we sought to establish the extent to which these difficulties were apparent in the (relatively) less abstract context of calculus theorems.

To learn about student strategies, ten students were interviewed. During these clinical interviews (Hunting, 1997), participants were asked to work through a version the survey. They were also asked to explain the reasons for their answers. Interviews were recorded using LiveScribe technology to capture both their written work and spoken answers.

Theorem: ***For all functions f, if f is differentiable at a point x = c, then f is also continuous at the point x = c.***

2) Suppose h is a function that is continuous at $x = 7$. Then
a. h is differentiable at $x = 7$.
b. h is not differentiable at $x = 7$.
c. not enough information to decide whether or not h is differentiable at $x = 7$.

Explain the reason for your answer:

Proposition: ***For integers a and b, if a§b then aba§bab.***

8) Suppose (7)(4)(7)§(4)(7)(4) is true.
Then 7§4 is
a. True.
b. False.
c. Not enough information to decide if True or False.

Explain the reason for your answer:

Figure 1. (Left) A sample task from Part I. (Right) A sample task from Part II.

Data Analysis

Survey responses were coded as "correct" or "incorrect." In addition to coding interviewees' responses as correct or incorrect, during the initial analysis of the interviews, notes were taken concerning the manner in which interviewees explained their answers. The focus was on the kinds of strategies participants used when working through the problems. This phase of the analysis was informed, in part, by prior research on student thinking about implication and additional rounds of analysis utilized techniques from Grounded Theory

(Strauss & Corbin, 1990) to further characterize student strategies. Categories and sub-categories were developed to characterize these strategies. This work builds off a previous work (Case, 2015) and the primary, new contribution in this report is a detailed analysis of the interviewee strategies for carrying out the tasks.

Survey Results

Consistent with prior research, students had difficulties with the abstract tasks. However, as Figure 2 shows, students were more successful on the calculus tasks than on the abstract tasks. On the calculus tasks, 63% answered at least three of the four tasks correctly and 33% answered all four correctly. In contrast, only 8% of students produced correct answers for at least three of the abstract tasks and none got all four correct. These differences between the calculus and abstract consistency percentages were statistically significant, suggesting that the context of calculus prompts students to engage differently with the calculus tasks than with the abstract tasks.

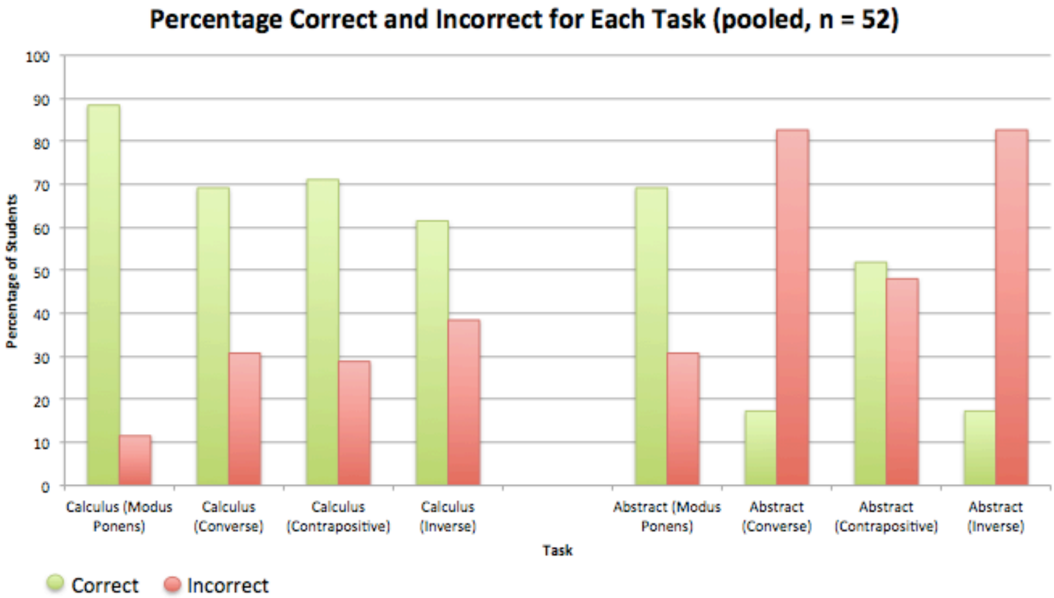

Figure 2. Student Performance on Calculus and Abstract Tasks from Survey Data.

We were also interested in potential relationships between success on one type of task and success on the other. For example, given that a student identified the correct answer to an abstract task, what is the conditional probability that they also answered the calculus version of that same task correctly? Given that a student did not correctly answer an abstract task, how likely are they to answer the calculus version of that same task correctly? The results (see Table 1) show that, for the *modus ponens*, *converse*, and *inverse* tasks, using a 2-proportion z-test, there was no statistically significant advantage when answering the calculus version of a task given a correct answer on the abstract version. However, for the *contrapositive* task, there does seem to be an advantage. Overall, these probabilities suggest that students who answer an abstract task correctly may not necessarily be more likely to answer the calculus version correctly. Stated differently, students can make sense of calculus theorems/definitions whether or not they are able to answer abstract logical reasoning tasks.

Interview Results

Although analyses of the survey data provided some insights (e.g., the calculus context seems to make some of the reasoning patterns easier for students to understand, the abstractly stated tasks are generally much more difficult for students, etc.), we wanted to understand more about student thinking concerning the inferences to gain further insight into the findings from the survey data analyses. From analysis of the interview data, we identified several different ways in which students approached the tasks. As displayed in Figure 3, there were three main ways of thinking (plus "other'), some of which had sub-categories that characterized the thinking at even finer levels of detail.

	Probability of Correct Calculus Answer Given a Correct Abstract Answer	Probability of Correct Calculus Answer Given an Incorrect Abstract Answer	p-value
Modus Ponens	.89	.88	$p > 0.05$
Converse	.89	.65	$p > 0.05$
Contrapositive	.85	.56	$0.01 < p < 0.05$*
Inverse	.56	.63	$p > 0.05$

Table 1. Conditional Probabilities of Answering Calculus Tasks Correctly (* indicates statistical significance with $\alpha = 0.05$)

We first consider the strategy located on the left-most branch. Interviewees who responded with "Child's Logic" (O'Brien, Shapiro, & Reali, 1971) tended to match truth-values (that is, they responded with "True" given a true premise and responded with "False" given a false premise). This strategy generates correct answers to two of the four tasks. Responses based on some formal knowledge of conditionals were also given a category. Here, participants explained their work by following some rule(s) (e.g., the converse of a conditional statement does not necessarily hold). Responses were also provided that involved the generation of examples. For example, some interviewees drew graphs or verbalized a particular mathematical scenario. Finally, some responses were difficult to categorize and/or did not seem to fit the previous three categories.

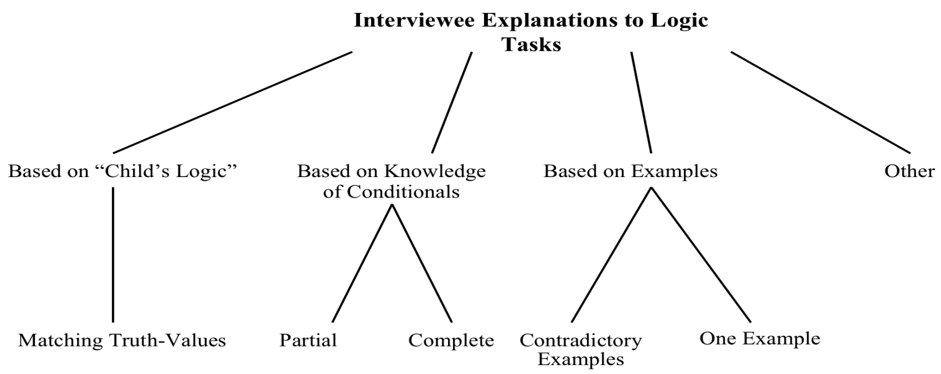

Figure 3. Types of Reasoning Exhibited by Interviewees.

Although each of the strategies provided insights into student thinking, here we discuss just one in detail. This strategy involves generating contradictory examples or situations in order to deduce the correct answer. This method was most often used on the calculus *converse* task and the calculus *inverse* task and it generated rich data on student thinking, and has potentially useful instructional implications (discussed later). We now examine a transcript excerpt that illustrates this kind thinking.

> **Jack:** *So, it's just like* [pauses to draw axes and says something inaudible] *and something goes like…this* [draws a continuous function with a sharp corner]. *And I mean you could define it as maybe two different line segments and try to do it that way, but the function itself isn't continuous* [we suspect, from the context, that he

meant “differentiable”] *because at that point there’s no specific, um, rate of change. However, for “b”, um…a function…very well could be not continuous and not differentiable. Say the function just* [draws a linear function with a hole]…*so you have some function that just has a hole in it. It’s not continuous and it’s not differentiable.*

Figure 4. Jack’s Contradictory Examples to the Calculus *Inverse Task.*

Here Jack produces two function graphs that invalidate two of the multiple-choice options (“*f* is continuous at the point” and “*f* is not continuous at the point”) in order to infer the correct answer: “not enough information to decide.” This strategy allows participants to take advantage of the familiar calculus materials presented in the problem so that the correct answer becomes clear. Five interviewees used examples at some point during the calculus portion of the interview. Four out of these five interviewees used contradictory examples.

On the abstract portion, only one student tried to answer a task with a generated example. As discussed above, survey participants did not perform as well on the abstract tasks. This may be in part because they are unable to create scenarios based on the task that they can work and reason with.

Implications and Further Avenues of Inquiry

Not surprisingly, our findings corroborate the established claim that students find abstract logic tasks challenging. However, in contrast, students responded to the calculus tasks in ways similar to how others have responded to logic tasks set in familiar contexts. In other words, although calculus ideas can be consider quite “abstract,” students manage calculus-based tasks in ways that suggest that the context enables them to reason more productively in comparison to the purely abstract tasks. On one hand, these results suggest that calculus students may need more preparation in formal logic, however, even without complete command over formal logic, they are still able reason appropriately when calculus ideas are involved and when they utilize “example generating” strategies. This suggests that it might be useful for instructors to help students develop this strategy. For example, when introducing a theorem such as “differentiability implies continuity”, instructors can model the “example generating” strategy while working through the various cases that might come up when faced with different functions. Some students may believe that they should just know answers to these kinds of tasks and by modeling how to reason through them with examples, instructors can strengthen students’ problem-solving skills. These findings also generated new questions for further research. It would be productive to investigate whether the wording of the theorem and theorem premise affect participant performance (for example, how would participants work through the four tasks if the given theorem structure resembled “if not A then B” rather than “if A then B?”). It might also be useful to examine the impact of instruction about “example generating” (and other) strategies on performance with the goal of enhancing students’ abilities to make sense of the theorems and definitions that are such an essential part of calculus. *Questions posed to the audience will include:* What other theorems or propositions might be worth examining in a study like this? Are there any other teaching implications that may be potentially derived from this study?

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