A framework for examining the 2-D and 3-D spatial skills needed for calculus

Nicole Engelke Infante
West Virginia University

Marjorie Darrah
West Virginia University

Kristen Murphy
West Virginia University

Having well developed spatial thinking skills is critical to success in many STEM fields such as engineering, chemistry, and physics; these skills are equally critical for success in mathematics. We present a framework for examining how spatial skills are manifested in math problems. We examine established spatial skills definitions and correlate them with the spatial skills needed to successfully solve a standard calculus problem – find the volume of a solid of revolution. This problem is deconstructed into steps and analyzed according to what 2-D and 3-D spatial skills are necessary to visualize and solve the problem.

Key words: spatial skills, calculus, rotation about an axis, volumes of revolution

Introduction

Mathematics, especially areas like geometry and calculus, require both 2-D and 3-D spatial thinking skills. Spatial thinking skills can be learned and, as expressed by the National Research Council (2006), should be taught at all levels of the education system. These same spatial thinking skills, once acquired, can be applied in many areas of science and mathematics. Calculus, in particular, presents many situations that require students to move between 2-D and 3-D representations, such as when they are to determine the volume of a solid of revolution.

Having well-developed spatial thinking skills is directly linked to future success in engineering careers (Adánez & Velasco, 2002; Miller, 1996; Sorby, 1999). In searching the literature, most programs that aim to increase the spatial thinking skills of students seemed to be targeted at engineers (Sorby, 1999). It is appropriate that there would be an emphasis in this area, but the authors argue that other majors, specifically STEM majors such as physics, chemistry and mathematics, also need these spatial thinking skills to be successful in their future careers and would benefit from similar skill building activities. We identified the spatial skills necessary to complete a common calculus problem – compute the volume of a solid of revolution. This problem was chosen as it requires spatial thinking skills that are known to be troublesome for students: rotations and cross-sections.

Here, we provide applicable spatial skills definitions (focusing on 2-D and 3-D spatial skills) used by several authors. Next, we deconstruct a classic second semester calculus problem, identifying the requisite spatial skills. From this deconstructed problem, we construct a framework for analyzing spatial skills required for calculus problems.

Spatial Skills Definitions

Pittalis and Christou (2010) and Cohen and Hegarty (2012) define two spatial skills important to the study of calculus: spatial visualization and spatial orientation. *Spatial visualization* is defined as the ability to comprehend imaginary movements in 3-D space or the ability to manipulate objects in imagination. An example of the use of this skill would be imagining the 3-D cube that can be created from a 2-D net with the six faces of the cube outlined in a plane. *Spatial orientation* is defined as the ability to remain unconfused by changing the orientation in which a spatial configuration is presented. An example of the use of this skill would be orienting disks or washers within the 3-D object when deciding how to compute its volume.

Along with these definitions, Pittalis and Christou (2010) further define four other skills that are applicable to calculus and apply to 2-D and 3-D representations. These skills are representing objects, structuring, measurement, and mathematical properties. *Representing objects* is defined as manipulating forms of 2-D or 3-D objects and constructing a 2-D or 3-D model. An example of the use of this skill would be constructing a 3-D shape by rotating a 2-D shape around an axis. *Structuring* is defined as constructing partitions of 2-D or 3-D objects and manipulating the partitioning of 2-D or 3-D objects. An example of the use of this skill would be using a cross section (disk or washer) to partition a 3-D shape to find its volume. *Measurement* is defined as calculating and estimating. An example of the use of this skill would be calculating or estimating the volume of a solid. The last of the four, *mathematical properties*, is defined as realizing, identifying, and comparing structural elements. Examples of the use of this skill would be finding intersection points, finding limits of integration, and realizing the interior and surface of the constructed 3-D object.

Although there are many other spatial skills defined, we have chosen the ones deemed most applicable to the study of calculus, specifically 2-D and 3-D representations. In the next section we examine one such classic calculus problem and identify what spatial skills are being used to move toward the solution at each step in the process.

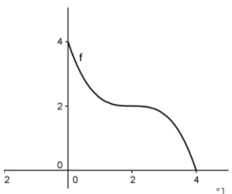
A Classic Calculus Problem

Below, we examine a solution to the problem:

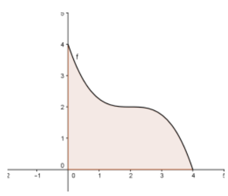
> Find the volume of the 3-D shape generated by rotating the region bounded by the function $f(x) = -\frac{1}{4}(x-2)^3 + 2$, the *x*-axis and the *y*-axis.

This is a typical second semester calculus problem. Upon close examination, it can be seen that there are many different spatial skills being used as a student proceeds toward a solution. Figure 1 presents the problem as a series of small steps that will later be classified using the suggested framework. While these steps are listed in the order a textbook or instructor might present them, they need not happen in this particular order. Rather, we are interested in capturing places where spatial skills might be useful to the problem solver.

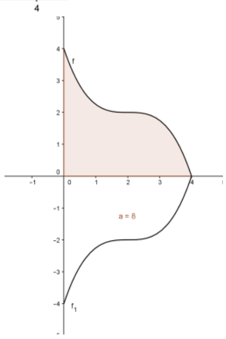
STEP 1:
Graph function: $f(x) = -\frac{1}{4}(x-2)^3 + 2$.



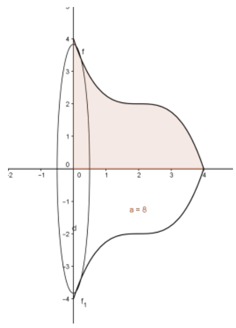
STEP 2:
Create 2D region with $f(x), x = 0, y = 0$.



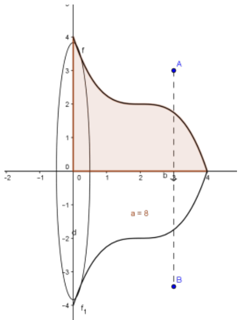
STEP 3:
Reflect the region about $y = 0$.



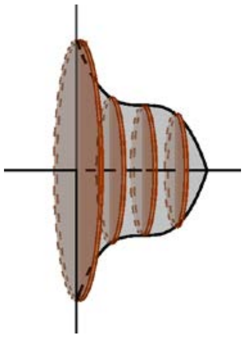
STEP 4:
Complete the 3D shape by rotating about the specified axis.



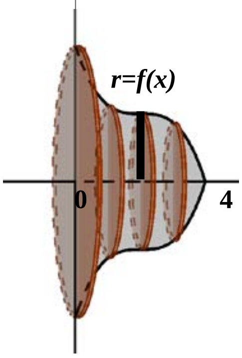
STEP 5:
Determine which way to slice:
horizontally or vertically?



STEP 6:
Position 3D disks.



STEP 7:
Draw radius and measure.
Determine limits of integration (0 to 4).



STEP 8:
Access volume of a cylinder formula:
$V = \pi r^2 h$

STEP 9:
Set up integral with boundaries.
$$\int_0^4 \pi (f(x))^2 dx = \int_0^0 \pi(-\frac{1}{4}(x-2)^3 + 2)^2 dx$$

*Figure 1***.** Spatial skills needed for rotating a region of the plane about an axis.

Proposed Framework: Spatial Skills for Calculus

Using the spatial skills definitions above, we constructed the framework in Figure 2 to classify the spatial skills needed to solve calculus problems. Figure 2 illustrates how the steps in the solution process in Figure 1 are mapped to one or more spatial skills.

	2 - D				***3 - D***			
	Representing	*Structuring*	*Measurement*	*Properties*	*Representing*	*Structuring*	*Measurement*	*Properties*
Visualization	Step 1: Draw 2D graph. Step 2: Identify region to be rotated.			Step 7: Determine the radius of a slice.	Step 4: Imagine the 2D region being rotated and the resulting 3D shape.	Step 6: Imagine slices filling up the figure.		
Orientation	Step 3: Identify the axis of rotation.		Step 7: Determine the radius of a slice.	Step 7: Determine the limits of integration.	Step 5: Determine which way to slice-horizontally or vertically?		Step 8: Determine the volume of a slice. Step 9: Set up and evaluate the integral.	Step 8: Identify the volume of a cylinder formula.

Figure 2. Framework to analyze spatial skills required in calculus problems. This figure illustrates the spatial skills required for the problem in Figure 1.

The row headings of visualization and orientation capture the broad spatial abilities of (1) being able to imagine and manipulate an object in 2-D or 3-D space, and (2) being able to remain unconfused when considering different perspectives of an object. There are two broad column headings of 2-D and 3-D, indicating that at various points in the problem solving process the student must think about either a 2-D object or a 3-D object. The four subskills – representing, structuring, measurement, and properties – identified by Pittalis and Christou (2010) may be required when thinking about either 2-D or 3-D objects and are represented by the columns under 2-D and 3-D respectively.

Note that each step of a problem may require more than one spatial skill. For example, step 7 requires 3 distinct spatial skills. First, the student must identify the radius by visualizing a particular property of a slice in a particular orientation. Then, the student needs to identify the range of appropriate values for that radius by accounting for the orientation of the stack of slices.

Using this framework to map other calculus problems that have spatial skills requirements (e.g., related rates, optimization, etc.) will allow us to identify which spatial skills are most used in the calculus curriculum and should get particular attention in remediation attempts. We now turn our attention to the spatial skills students possess when entering calculus.

Discussion

Although some programs have provided avenues for engineering students to improve their spatial thinking skills, there is a lack of attention to the development of spatial thinking skills for other majors. We need to promote students' understanding of geometric concepts and properties beyond using an algorithm or formula to get an answer. There are many places in the calculus curriculum where a lack of spatial skills hinders the understanding of concepts. One of the largest obstacles to college success is that students are arriving unprepared for the rigors of the college math curriculum; in particular, more than 40% and as many as 75% of students entering college place into a developmental math course (Twigg, 2013). While there are mechanisms in place for the development of other skills, such as arithmetic, algebra and geometry, not many colleges are addressing spatial thinking skills.

Further refinement of the framework for classifying calculus problems will allow us to analyze more of the curriculum to determine appropriate diagnostics and interventions. Considering the findings from the analyses of the curriculum, we will choose diagnostic spatial skills tests that align with the skills needed. Two examples of possible tests are the Purdue Spatial Visualization Tests: Visualization of Rotations (PSVT:R) (Guay, 1976;Yoon, 2011) and the Santa Barbara Solids Test (SBST) (Cohen and Hegarty, 2012).

The next step in this project is to observe the development of a group of 10-12 first semester calculus students' spatial thinking skills. We will obtain a baseline by using the PSVT:R and SBST tests and conducting semi-structured interviews with each student. The students will then be interviewed periodically over the course of several semesters as they encounter problems involving spatial thinking skills to identify how their spatial skills develop.

This work is the first step toward understanding what spatial thinking skills students have when entering calculus and determining how we can better understand where they need to improve. From the findings, we hope to pinpoint areas in the curriculum that rely on spatial thinking skills and determine if students have those necessary skills. We intend to develop interesting activities, both non-technology and technology-based, that could be used at critical points in the curriculum to assist students in developing and refining these critical spatial thinking skills. Much like the program developed by Sorby (1999) for engineering students, our goal is to determine which calculus students may have problems and with what concepts, and then provide additional activities to assist those students in the further development of their spatial skills.

We are interested in receiving feedback from the audience on the following questions:

1. What spatial skills are critical for success in mathematics that we have not captured?
2. What categories of problems should we examine next?
3. What ideas do you have for activities to build spatial skills?

References

Adánez, G. P. , & Velasco, A. D. . (2002). Predicting academic success of engineering students in technical drawing from visualization test scores. *Journal of Geometry and Graphics, 6*(1), 99-109.

Cohen, C. A., & Hegarty, M. (2012). Inferring cross sections of 3D objects: A new spatial thinking test. *Learning and Individual Differences, 22*(6), 868-874. doi: 10.1016/j.lindif.2012.05.007

Guay, R. B. (1976). Purdue spatial visualization test. West Lafayette, Indiana: Purdue Research Foundation.

Martín-Gutiérrez, J., Saorín, J. L., Contero, M., Alcañiz, M., Pérez-López, D. C., & Ortega, M. . (2010). Design and validation of an augmented book for spatial abilities development in engineering students. *Computers & Graphics, 34*, 77-91. doi: 10.1016/j.cag.2009.11.003

Miller, C. A. (1996). Historical review of applied and theoretical spatial visualization publications in engineering graphics. *Engineering Design Graphics Journal, 60*(3), 12-33.

National Research Council. (2006). *Learning to think spatially*. National Academic Press, Washington, DC.

Pittalis, M. , & Christou, C. . (2010). Types of reasoning in 3D geometry thinking and their relation with spatial ability. *Educational Studies in Mathematics*, *75*(2), 191-212.

Sorby, S. A. (1999). Developing 3-D spatial visualization skills. *The Engineering Design Graphics Journal, 63*(2), 21-32.

Sorby, S. A., Casey, B., Veurink, N., & Dulaney, A. (2013). The role of spatial training in improving spatial and calculus performance in engineering students. *Learning and Individual Differences, 26*, 20-29.

Twigg, C. (2013). Improving Learning and Reducing Costs: Outcomes from Changing the Equation. *Change: The Magazine of Higher Learning*, *45*(4), 6-14.

Yoon, S. Y. (2011). *Psychometric properties of the Revised Purdue Spatial Visualization Tests: Visualization of Rotations (the Revised PSVT:R).* (Ph. D.), Purdue University.