Ways of understanding and ways of thinking in using the derivative concept in applied (non-kinematic) contexts

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Much research on students' understanding of derivatives in applied contexts has been done in kinematics-based contexts (i.e. position, velocity, acceleration). However, given the wide range of applied derivatives in other fields of study that are not based on kinematics, this study focuses on how students interpret and reason about applied derivatives in non-kinematics contexts. Three main ways of understanding or ways of thinking are described in this paper, including (1) invoking time, (2) overgeneralization of implicit differentiation, and (3) confusion between derivative expression and original formula.

Key words: calculus, derivative, applications, ways of understanding and thinking

The calculus concept of the derivative is important both within mathematics and in other disciplines like physics, engineering, economics, biology, and statistics. As such, researchers have been interested in how students use the derivative in a range of contexts (e.g., Bucy, Thompson, & Mountcastle, 2007; Christensen & Thompson, 2012; Zandieh, 2000). However, much of the mathematics education research dealing with applications of the derivative has been centered on the kinematics applications of position, velocity, and acceleration (e.g., Berry & Nyman, 2003; Marrongelle, 2004; Petersen, Enoch, & Noll, 2014; Schwalbach & Dosemagen, 2000). While velocity and acceleration are certainly common and useful applications of the derivative, there are myriad other uses of this concept in fields of study outside of mathematics. Given this deficit in exploring student understanding of the derivative in a wider variety of applications, I focus this paper on how students interpreted and reasoned about the derivative concept in applied, *non-kinematics* contexts. Specifically, I relate certain *ways of understanding* and *ways of thinking* exhibited by students that seemed particular to working with applied, non-kinematics derivatives.

Ways of Understanding, Ways of Thinking

For this paper, I draw on the constructs of *ways of understanding* and *ways of thinking* (Harel, 2008; Harel & Sowder, 2005) to explore certain aspects of how students might think and reason about derivatives in applied, non-kinematics contexts. First, Harel and Sowder (2005) use the term "mental act" to denote any internal mental action, such as interpreting, inferring, explaining, or searching. One single cognitive product of a mental act (or acts) in one given situation is termed a *way of understanding*. For example, if a student sees "*dy/dx*" and thinks "that's the slope," then the student has produced a single way of understanding *dy/dx* through the mental act *interpret derivative symbol*. If, in observing a student, a particular characteristic is found to be repeatedly associated with a given mental act, then Harel and Sowder term that a *way of thinking*. In the example, if the same student indicates in many situations or problem contexts that a derivative is a "slope," then that student is considered to have a *way of thinking* associated with interpreting the derivative.

The constructs of ways of understanding and ways of thinking are used in this paper to explore idiosyncrasies and difficulties evidenced by students in thinking about derivatives in applied, non-kinematics contexts. Some of the idiosyncrasies, which are discussed as possible ways of thinking, seemed specific to the applied context of the derivatives. Some of the difficulties appeared to be particular ways of understanding produced by the students as they performed mental acts related to reasoning about derivatives in applied contexts.

Interview and Survey Data

The data used for this paper consists of hour-long, task-based interviews with six first-semester calculus students, and surveys conducted with 38 first-semester calculus students. The six interviewed students were recruited at the end of the same first-semester calculus class, which was taught in a fairly "traditional" manner by a mathematics department faculty member at a large university in the United States. Here I use "traditional" simply to indicate that nothing seemed unusual in the presentation of the material in this course. In this paper, the interviewed students are given the pseudonyms: Jack, Lily, Noah, Zoe, Oliver, and Toby.

The 38 surveyed students came from two different calculus classes at the same university (25 students in one class, 13 in the other), with these two classes being different from the one in which the interviewed students were recruited. Thus, students were recruited from a total of three different classes with three different instructors at the same university. The classes were all taught in what could be described as a fairly typical manner.

The six interviewed students were given a range of contexts in which they were asked to discuss the derivative concept. The interview consisted of five prompts (see below), though for the purposes of this paper I focus only on the three "applied" prompts, which asked the students to calculate and discuss derivatives in applied, non-kinematics contexts. Note that all "fractional" expressions, like "df/dx," are formatted this way for the purposes of the paper, but were given to the students as "$\frac{df}{dx}$."

1. Let $f(x) = x^4$. Calculate df/dx and explain what it means.
2. Given the formula $z = rt + st^2 + rs/t$, calculate dz/dt and explain how you did it.
3. Suppose we have a cylinder with radius r and height h [an image of an unlabeled cylinder is provided]. The volume formula for a cylinder is $V = \pi r^2 h$. (a) Calculate dV/dr. What does this answer tell you? (b) Calculate dV/dh. What does this answer tell you?
4. The force of gravity (F) is dependent on how far an object is from the Earth's center (r), given by the formula $F = GmM/r^2$. (M and m are the mass of the earth and the object and G is the "gravitational constant.") (a) Calculate dF/dr. What does that tell you? (b) Calculate dF/dm. What does this answer tell you?
5. What would these following derivatives tell you? Should they each be positive or negative? (a) dS/dp, if p = price of a book, and S = number of books sold; (b) dV/dr, if V = volume, and r = radius of a sphere; (c) dM/dt, if M = memory, and t = time.

Many follow-up questions were used during the interviews based on the students' responses, such as "Why does your answer tell you that?," "What does it mean that the answer has a negative sign?," or "For every increase in __, will I get the same change in __?"

The interviews were fully completed before the administration of the surveys. This was done intentionally to allow a preliminary analysis of the interview data to occur prior to the survey creation. In this way, I identified individual students' potential ways of understanding and ways of thinking from the interview data and then tailored the survey questions to see if some of those same ways of understanding/thinking might be replicated by a larger sample of students. In order to create a brief survey protocol, the surveys only contained two applied, non-kinematics questions, which corresponded to the third and fourth interview prompts:

1. The volume of a cylinder with radius *r* and height *h* is given by $V = \pi r^2 h$. (a) Calculate *dV/dr* and then state the meaning of *dV/dr*. (b) Suppose that *r* increases. Describe as much as you can what your answer to part (a) tells you about the cylinder's volume. *How* does your answer tell you that?
2. The force of gravity between an object and the Earth is given by $F = GmM/r^2$ (*r* is the object's distance from Earth's center, *M* and *m* are the masses of the earth and the object, and *G* is a constant). (a) Calculate *dF/dr* and then state the meaning of *dF/dr*. (b) Suppose that *r* increases. Describe as much as you can what your answer to part (a) tells you about the force of gravity. *How* does your answer tell you that?

Data Analysis

Since much work has already been done in examining how students understand the fundamental ideas contained in the derivative concept (e.g., Habre & Abboud, 2006; Orton, 1983; Zandieh, 2000), this paper is not meant to repeat the results of these prior studies. Consequently, I did not focus the analysis for this study on students' overall understandings and meanings assigned to the generic derivative concept. Rather, I focused on exploring aspects of students' thinking and reasoning that seemed pedagogically important regarding working with applied, non-kinematics derivatives.

The preliminary analysis of the interview data, mentioned in the previous section, consisted of using open coding (Strauss & Corbin, 1998) to identify plausible ways of understanding/thinking specific to applied, non-kinematics derivatives exhibited by the individual students. This led to the creation of three main categories, which are described in the results section: (1) invoking time, (2) overgeneralization of implicit differentiation, and (3) confusion between the derivative expression and the original formula. Once this preliminary analysis had been conducted, the survey protocol was created and administered to identify whether these categories would be observed in a larger sample.

Following the survey administration, a more systematic coding of the data occurred by going through the interviews and surveys to code for all instances of the three categories. Throughout the process, I remained open to the possibility of new categories emerging. While no "top-level" categories were introduced at this stage, a distinct subset of the third category took shape that centered on confusion around applied derivative expressions that were constant. The data was re-coded a final time looking for instances of this subcategory. Unfortunately, this subcategory was explicitly identified *after* the survey administration, meaning no question had been included on the survey to target it in the larger survey sample.

Results

In this section, the three categories listed in the previous section are discussed through the lens of ways of understanding and ways of thinking. That is, there appeared to be certain idiosyncratic tendencies from many of the students, which provided evidence of ways of thinking related to applied derivatives. In addition, a common difficulty became evident in terms of how students interpreted applied derivatives. While perhaps not a way of *thinking*, it appears to be a common way of *understanding*.

Invoking time

To preface this subsection, I wish to draw attention to the fact that none of the "applied" interview prompts (with the exception of 5c) and none of the survey prompts explicitly required *time* as a factor in the derivative. For example, the derivative *dV/dr* does not require *r* nor *V* to change quickly or slowly in time, nor even at a steady rate with respect to time. It is therefore interesting that four of the interviewed students and 14 of the surveyed students interjected time explicitly into the contexts as they calculated and explained the applied derivatives, as demonstrated by these examples:

Lily: [Explaining *dV/dr*] Like say [*r*] is changing at a rate of one meter per second, that's really fast, but if it's getting bigger constantly, this is going to, the volume itself… if it's one meter per second… it changes smaller at first, but then bigger.

Noah: [Explaining *dV/dh*] If we're increasing the height by one every time, assuming that it happens in one-second or, like, the next time interval, the next time, then that would just be the same relationship. So, it would increase at the same, at a constant rate.

Survey: [Explaining *dF/dr*] It's talking about the force in relation to distance. It's related to time and mass.

Survey: [Explaining *dV/dr*] Volume is changing in relation to *r* in time.

Thus, for many students the mental act "describe" or "explain" applied derivatives produced a way of understanding that explicitly attended to time. I hasten to add that involving time is *not incorrect*, since changes in real-world quantities can essentially only be envisioned over time. Furthermore, for many of the students, it seemed that interjecting time was a useful way to explain the meaning of these derivatives. For example, in Lily's excerpt, she used the context of a radius increasing at a steady rate in time to help explain that the volume would always grow, but by a smaller rate at first and then by a larger rate later.

While these could be characterized as ways of understanding, since they are stand-alone explanations, some interviewed students had a strong tendency to insert time into most of the problem contexts, as exemplified by Zoe:

Zoe: [Explaining *dV/dr*] If it was normal, let's say it's normal, it would be *dV/dt*, which would mean we would have meters cubed divided by time, in seconds. [Attempts to use analogous reasoning to interpret *dV/dr*, but unsuccessfully.]

Zoe: [Explaining *dS/dp*] As the price gets cheaper, the number of books sold would decrease. That doesn't, well, it depends [trails off]. But I suppose over time, if it's a cheaper price for a longer amount of time, it would increase [*S*].

Interviewer: [Regarding prompt 5b] Why would the values of [*V*] be getting bigger?
Zoe: I don't know [pause]. Alright, [*dV/dr*] means change in volume over change in radius, so [long pause]. The rate at which—there's no time involved!… So, as we're changing the radius, imagine the radius is time, because as you're affecting the radius, you can't do it without time, because you can't do things outside of time… If we negate the middle-man and negate the change in radius, then we'd just have the change in volume as the time changes.

Zoe's repeated inclusion of time shows that these were more than ways of understanding, but together demonstrate a strong way of thinking. Whereas some students, like Lily, could use time effectively to imagine a non-time-based derivative as needed, Zoe's way of thinking seemed to hinder her reasoning at times, becoming more of a crutch than an aid. She often desired to alter the nature of the derivative from one that is time-less to one that is based on time. In the last excerpt, she even cut the radius from the context altogether in order to bring the derivative in line with her strong time-dependent thinking.

Overgeneralization of implicit differentiation

The second category I discuss in this paper is less conceptual in nature and represents more of an overgeneralization of a specific class of derivative problems. In typical first-semester calculus courses students study applications of the derivative including optimization problems and related rates. Related rates deal with implicitly defining variables in terms of another "latent" variable, requiring implicit differentiation to solve the problems. For example, in $V = {}^4\!/_3\pi r^3$, the volume and radius could be thought of as functions of temperature (say, if the sphere is metallic), leading to $V(T) = {}^4\!/_3\pi[r(T)]^3$. Then derivatives such as dV/dT or dr/dT could be calculated through implicit differentiation. In this study, four of the interviewed students and 27 of the surveyed students assumed some of the variables in the formulas to be implicitly defined in terms of either the variable of differentiation or some other variable. For example, many students seemed to think that a derivative such as dV/dr required all or some variables to become implicitly defined in *r*—sometimes even the variable *r* itself! Time was also often invoked as a latent variable, making this category connected, in part, to the previous category. The following are examples from the students' work (note that not all calculations would represent correctly calculated derivatives):

- $dV/dr = 2\pi r \, {}^{dh}\!/_{dr}$
- $dV/dr = 2\pi rh \, {}^{dr}\!/_{dr}$
- $dV/dr = 2\pi rh \, {}^{dr}\!/_{dt}$
- $dV/dr = 2\pi rh + \pi r^2 \, {}^{dh}\!/_{dt}$
- $dV/dt = \pi r^2 \, {}^{dh}\!/_{dt} + 2\pi rh \, {}^{dr}\!/_{dt}$
- $dF/dr = G \, {}^{dm}\!/_{dr} M + Gm \, {}^{dM}\!/_{dr}$
- $dF/dr = [r^2 G \, {}^{dm}\!/_{dr} \, {}^{dM}\!/_{dr} - GmM(2r)]/r^4$
- $dF/dr = [r^2(Gm \, {}^{dM}\!/_{dt} + MG \, {}^{dm}\!/_{dt}) - (GmM)2r \, {}^{dr}\!/_{dt}]/r^4$

I once again note that implicitly defining some variables in terms of others is not necessarily incorrect, though many of the ways in which students did so in this study could be considered incorrect. For example, in $V = \pi r^2 h$, unless an extra condition is placed on the relationship between the radius and the height, height is not a function of the radius at all.

Most of the students who forced some variables to be implicitly defined in terms of others did so for more than one problem. For example, most surveyed students who did this did so on both problems. As such, I consider this category to represent a way of thinking for many students. While more procedural in nature, it is important for educators to be aware of this tendency, given that over two-thirds of the students in this study forced implicit differentiation onto the non-implicitly-based applied derivatives.

Confusion between the derivative expression and the original formula

Perhaps the most important category to discuss in this paper is the confusion many of the students exhibited at times between the derivative formula and the original formula. Five of the interviewed students and 16 of the surveyed students gave evidence of this type of confusion, which is nearly half of all the students in this study. When interpreting and making sense of the derivative formula they calculated, many students began to explain the derivative formula as though it provided a value for the *original* quantity of interest. The following excerpts are examples of this type of confusion.

Oliver: [After correctly calculating $dF/dm = GM/r^2$] It's the change in force as we change the mass of the object. And what this is telling us is, because there is no mass [little *m*] in the equation that the force isn't subject to the mass of the object.
Interviewer: OK, so would that mean that I could make the object more massive or less massive and that has no effect on force?
Oliver: Right.

Lily: [Explaining whether dS/dp would be positive or negative] I guess that would be positive, because there's no such thing as selling a negative number of books.
Lily: [Discussing what it would mean if $dS/dp = 0$] Zero would mean that no books are being sold at that specific price.

Zoe: [Explaining $dV/dh = \pi r^2$] This would imply that there is no change!
…
Interviewer: Whether you feel like it makes intuitive sense or not, what do you feel like that [points to the derivative formula] should be telling you?
Zoe: That no matter how *h* changes, *V* remains the same. That's doesn't make sense!

Survey: [Explaining $dV/dr = 2\pi rh$] The rate at which the volume of the cylinder is increasing is 2x the rate at which the radius is increasing (π and *h* are constants). Answer tells me that, because the derivative of the function is $2r$ (π and *h* being constants). Thus I know that the rate at which the volume is increasing is double the rate the radius is increasing.

In these examples, it is clear that the students had essentially read the derivative expression as directly providing the value of the original quantity whose derivative was being calculated. In other words, the students explanations would make sense if the following substitutions were made:

- $dF/dm = GM/r^2 \rightarrow F = GM/r^2$
- $dS/dp \rightarrow S$
- $dV/dh = \pi r^2 \rightarrow V = \pi r^2$
- $dV/dr = 2\pi rh \rightarrow V = 2\pi rh$

Essentially, these students seemed to have a particular way of understanding equations, in these moments, in which an equation takes on the meaning [quantity] = [expression]. That is, regardless of what is on the left of the equation, whether *F* or dF/dr, it seems to mean "quantity" instead of other possibilities, like "rate of change." I wish to point out that many of the students were *not* consistent in doing this, but that they only occasionally made this error. As such, I am careful not to call it a way of thinking, which would assume a greater regularity than was visible in the data. Rather, for most students, I see it as a way of understanding, since it was the result of a particular mental act at one point in time. Even so, this conceptual mistake happened so often with both the interviewed and the surveyed students that this way of understanding seems to be an issue educators should be aware of.

Confusion with applied derivatives that are constant

In discussing the confusion between the derivative expression and the original expression, I note that the most frequent context for this confusion during the interviews was when the applied derivative yielded an expression that was *constant*. (Note that since I did not provide a constant derivative on the survey, I cannot comment about this issue for the surveyed students.) In expressions like $dV/dh = \pi r^2$ and $dF/dm = GM/r^2$, the variable of differentiation is not present on the right side of the equation. These cases would indicate a constant *rate of change*, which many of the students overlooked, believing it to mean a constant *quantity* instead. This led to much frustration in the students as they struggled to identify *why* the variable, such as h or m, would not have an impact on the quantity V or F. Some students, like Oliver and Zoe, were never able to reconcile the discrepancy between what the derivative expression seemed to be saying and what they intuitively believed to be true. It is important to note that these same students, during the pure-mathematics prompts, showed no difficulty whatsoever in making sense of a constant derivative in pure mathematics contexts (the interviewer asked about this as a follow up to prompt 1). As such, it seems that there was something fundamental about the applied nature of the derivatives used in this study that prevented the students from accessing resources they certainly had about the meaning of constant derivatives in pure mathematics contexts.

Discussion and Implications

In this paper I have highlighted three pedagogically important ways of understanding or ways of thinking exhibited by many of the interviewed and surveyed students in the study. *Invoking time* seemed to be a useful way of understanding for some students, though problematic for others when it became an almost uncontrollable way of thinking. This suggests that it would be important for calculus instructors to have explicit discussions regarding *time* and how it comes into play with applied, non-kinematics derivatives. Since typical applications of the derivative, including velocity and acceleration, *are* time-based, it may be important to explore other, non-time-based derivatives during instruction as well.

Overgeneralization of implicit differentiation was the most commonly observed of the three categories in this study. This suggests that many students, when learning certain types of applications, such as related rates, may overgeneralize the implicit differentiation procedure into a belief that non-kinematics-based applied derivatives require variables to be defined implicitly with respect to either the variable of differentiation or time. While in some cases this may be fine, in many cases it may be incorrect, or at the least very burdensome. Thus, calculus instructors may need to have meta-discussions on the types of applications studied in class, so that students do not mistakenly believe that those procedures must be used for *all* applied problems.

Perhaps the most important conceptual difficulty students had was in *confusing the derivative expression with the original expression*. This seems similar to what Musgrave and Thompson (2014) call "function notation as idiom," wherein the symbol on the left of the equation just represents a "name" for the equation, leading to potentially problematic expressions like "$f(x) = n(n\text{-}1)/2$" (p. 283). In other words, students might not pay careful attention to *what* exactly is on the left side of the equation, but may rather simply view it as a label, usually for a quantity's value (as opposed to other possibilities, like a rate of change). The right side of the equation is "where the math happens" (Musgrave & Thompson, 2014, p. 286), and the expression's value tends to represent the *magnitude* of the quantity of interest.

Overall, this study shows that there are additional conceptual and procedural layers to working with applied derivatives in *non-kinematics* contexts. As such, perhaps the extensive emphasis placed on kinematics examples in calculus (Berry & Nyman, 2003; Marrongelle, 2004; Schwalbach & Dosemagen, 2000) may not be adequately developing the resources needed to work with and reason about non-kinematics derivatives. Since there is a significant range of applied derivatives in other fields of study that are *not* based on kinematics, or even on time, it may be important for calculus educators to bring in these types of examples more regularly during calculus instruction. Doing so may help students develop the conceptual and procedural resources to effectively use and reason about these types of derivatives.

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