Students' conceptions of factorials prior to and within combinatorial contexts

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Counting problems offer rich opportunities for students to engage in mathematical thinking, but they can be difficult for students to solve. In this paper, we present a study that examines student thinking about one concept within counting, factorials, which are a key aspect of many combinatorial ideas. In an effort to better understand students' conceptions of factorials, we conducted interviews with 20 undergraduate students. We present a key distinction between computational versus combinatorial conceptions, and we explore three aspects of data that shed light on students' conceptions (their initial characterizations, their definitions of 0!, and their responses to Likert-response questions). We present implications this may have for mathematics educators both within and separate from combinatorics, and we discuss possible directions for future research.

Keywords: Combinatorics, Discrete mathematics, Factorials, Counting

Introduction and Motivation

Counting problems provide opportunities for interacting with problems that are easy to state and understand, but that require deep and non-algorithmic mathematical thinking. Brualdi says the following about counting problems: "The solutions of combinatorial problems often require *ad hoc* arguments sometimes coupled with use of general theory. One cannot always fall back onto application of formulas or known results" (2004, p. 3). In this paper, we explore one specific concept within counting that we feel plays an important role in combinatorial enumeration: factorials. The factorial of a natural number n is defined as the product of the first n natural numbers (for instance, Epp (2010) defines n factorial as follows: "For each positive integer n, the quantity n factorial denoted $n!$, is defined to be the product of all the integers from 1 to n" (p. 181)). Factorials themselves are defined and exist in contexts outside of a combinatorial setting, and yet they play a significant role in the solving of counting problems – both due to the fact that they can be interpreted as having inherent combinatorial meaning and because they are a basic component of many fundamental counting formulas. Because of this, we are interested in learning more about what conceptions students have of factorials within and without the counting context, and we seek to better understand how students' pre-existing conceptions of factorials (even prior to reasoning about them in a combinatorial context) might interact with their combinatorial thinking about factorials. We seek to answer the following research question: *How might we characterize students' initial conceptions about factorials, and how do such conceptions interact with students' solving of counting problems?*

Literature Review and Theoretical Perspectives

Counting problems have been shown to be difficult for students at a variety of levels. This is seen both in low success rates (e.g., Eizenberg & Zaslavsky, 2004) and in qualitative evidence of student struggles (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Hadar & Hadass, 1981). Although difficulties persist, there has also been growing evidence of students' success with counting problems. This has included identifying beneficial ways of thinking about counting problems (e.g., Lockwood, 2014; Halani, 2012; Maher, Powell, & Uptegrove, 2011), identifying and testing the value of particular instructional interventions (Lockwood, Swinyard, & Caughman, 2015; Mamona-Downs & Downs, 2004), and developing models and schemes of students' combinatorial thinking (Lockwood, 2013; Tillema, 2013). One possibility toward continued understanding of the state of students' combinatorial reasoning is to look at a very specific yet fundamental concept related to counting.

In this paper, we focus on the concept of factorials because it is itself a foundational combinatorial idea with a widely applied formula, and it is a key aspect of many other combinatorial concepts (and their formulas) that students encounter as they solve counting problems. Though we did not identify any research explicitly about students' reasoning about factorials, there has been a tradition, beginning with Piaget and Inhelder (1975), of closely examining students' mental processes involved in reasoning about particular concepts. We are also motived by a recent study (Lockwood, et al., 2015) in which students seemed to have preconceived notions of factorials that were affecting their reasoning about counting problems. We seek to better understand if other students hold similar preconceptions, and also to consider if and how certain conceptions come into play as students reason about factorials in counting contexts. We take conceptions to mean a students' mathematical understanding about a particular idea.

The study is framed within Lockwood's (2013) model of students' combinatorial thinking. This model proposes three components of students' combinatorial thinking and elaborates the relationships between these components. *Formulas/Expressions* are mathematical expressions that yield a numerical value, while *Counting Processes* refer to the processes in which a counter engages (either mentally or physically) as they solve a counting problem. Finally, *Sets of Outcomes* consist of the collection of objects being counted – those sets of elements that are generated or enumerated by a counting process (p. 252 – 253). In this paper, we use the model as a way of framing students' combinatorial reasoning about a particular combinatorial construct.

Methods

Participants. The participants in this study were 20 undergraduate students at a large university in the western United States. Fourteen of the students were taking a calculus course at the time of the interviews, and six were taking 300-level mathematics courses that came after discrete mathematics, including advanced calculus, topology, linear algebra, probability, and numerical analysis. We targeted these two groups of students because we wanted to include in our sample both some students who had taken discrete mathematics at the university level and some who had not. There were fourteen male students (ten in calculus, four in the advanced courses) and six female students (four in calculus, two in advanced courses).

Data collection. The design of our study was to conduct individual, semi-structured, task-based interviews (Clement, 2000; Goldin, 1997), and we used Livescribe pens to record the interviews, which allowed us to capture what students wrote and said in real time. We created a list of symbols commonly seen in college-level mathematics classes, and we asked the students to indicate which symbols they recognized. If the students recognized the factorial symbol, we proceeded to ask them a line of questioning to probe their understanding of factorial. We then asked them a handful of initial questions about factorials, including statements that required a response on a Likert scale. We concluded the interview by giving the students counting problems one at a time, asking them to verbally explain their reasoning and write down their thought processes as they went. If students did not recognize the factorial symbol, or recognized it and gave an incorrect interpretation of the symbol, we immediately gave them the counting problems, returning to the reflective questions and Likert response statements if they showed any subsequent sign of having remembered what the factorial symbol meant. Because students' responses and background varied considerably, and because the interviews were semi-structured, there is a variety among the specific tasks and counting problems each student completed.

Data Analysis. After conducting and recording the interviews, we re-listened to the interviews and watched the files outputted by the Livescribe pens. We created content logs of the interviews, which provided a detailed, time-stamped description of what happened and included transcriptions of particularly noteworthy episodes. The research team discussed and coded the students' responses to the initial questions, and the data revealed emergent dimensions of students' conceptions of factorial, which we discussed in relationship to Lockwood's (2013) model of combinatorial thinking. Next, we studied the Likert responses by first calculating the average and standard deviation of the responses for each question. We also discussed responses that were particularly surprising and attempted to determine and articulate why students might have responded in unexpected ways. Finally, each author individually coded student responses to the counting problems, looking for particular factors of interest for each problem. Any discrepancies in coding were addressed via discussion among the research team.

Results

In presenting the results, we focus on sharing empirical evidence of students' initial conceptions of factorials, as seen in their responses to initial questions and to the Likert-response questions. Due to space we do not report on their work on the counting problems.

Computational and Combinatorial Conceptions of Factorials

A major finding about students' reasoning about factorials is that there is a fundamental distinction between two conceptions of factorials: computational and combinatorial. We do not claim that is it mathematically a new insight. However, this distinction arose in our data in several ways, and we share it as a finding because it seemed to reflect an important difference in conceptions for our students. By a *computational* conception, we mean that students think of n factorial in terms of its numerical definition as the product of the first n positive whole numbers. A student with a computational conception might be able to use and manipulate factorials in expressions or equations, as strictly a numerical calculation. A *combinatorial* conception involves an understanding that factorials have some intrinsically combinatorial meaning – specifically as being related to the number of ways of arranging n distinct objects. Someone with a combinatorial conception may think of $n!$ as the number of arrangements of n distinct objects, and they additionally may be able to conceive of a process by which the product $n!$ can generate all of the arrangements of n things. With a combinatorial conception, there is a natural way to relate factorial to some combinatorial meaning and not only as a computable expression. We provide evidence of these two conceptions in students' initial characterizations and in their definition of $0!$.

Students' Initial Characterizations of Factorial. For each student who had recognized the factorial symbol, we asked them what they thought it meant and how they would explain it to someone else. Of the 20 students, 15 had seen the factorial symbol before and could provide a statement of it. Of those 15, 12 of the students gave a correct definition, with all of these students expressing factorial in terms of its computational definition. Eight of them defined $n!$ as a decreasing product, such as Student 6 who replied, "You take whatever number n is and you multiply it by all whole numbers fewer than it, stopping at 1." The remaining four students who provided a correct definition described n! as an increasing product. For example, Student 17 said, "You're multiplying all the integers together in order until the n."

In addition to providing correct computational definitions, there were two students (1 and 8) that additionally defined factorial combinatorially. Student 1 defined factorials as follows: "It's called n factorial, and it gives us the number of possibilities we can arrange things in order. Like, if we have n distinct objects, and we'd like to put them in a certain order, we'll have n factorial which is n times n-1 times dot dot dot times 1." Similarly, Student 8 gave his initial computational definition by saying "I would just say it's n times n – 1 times n – 2 all the way down to 1. I would, you know, explain it recursively." When asked if he could think of more than one way to explain a factorial to someone else, he responded,

Student 8: "Hmm. I guess if you're talking about, like permutations of, yeah, if you're talking about permutations of, like, 8 objects or something, you'd say, okay so for the first one I have 8 choices, then seven choices, then six choices, then five choices, and explain that and say 'Oh, and this is really annoying to write it out, so we'll call it the factorial function.'"

The three students who incorrectly characterized factorial did so as involving addition instead of multiplication, which suggests perhaps that these students were familiar with the notion of factorial but did not have a solid understanding of it. As an example, Student 20 said, "Um, I think it was just, um, to notate the fact that, um, uh, it repeat, well, the symbol is, like, like, 5 exclamation point is like 5+4+3+2+1, isn't it? And I forget why we needed to use that." Although only two students defined factorial combinatorially here, a number of students also brought up combinatorial interpretations of n! in other parts of the interviews, such as when they were asked to solve counting problems or when justifying why 0! is 1. This suggests that while most of the students had existing computational conceptions of factorial, some had a sense that factorials could apply in a combinatorial setting.

Student definitions of 0!. We also saw evidence of computational versus combinatorial conceptions in students' discussions of how 0! should be defined. We asked 12 students about how they understood 0!, and nine students responded (the other students did not have a guess). We note that the convention of having 0! defined as 1 is easily justified by the combinatorial characterization of factorial, because there is only one way to arrange no objects. The computational justification for why 0! = 1 is related to the convention of empty products (the product of no factors) equaling the multiplicative identity, which is 1.

Although there exists a computational, non-combinatorial justification for why 0! might be 1 (the empty product convention) the students who did not provide a combinatorial justification were not able to articulate this argument – indeed, many of the reasons the students provided are not correct or convincing. For example, Student 9 said 0! would be 0 and explained, "I'm not sure if this is the technical definition, but n factorial is all the numbers from n to 1, and so if n was zero, then it'd be like zero times nothing." In fact, the students who did not reason combinatorially about 0! did not provide reasonable justifications about why 0! is 1. These results suggests that students' computationally knew how 0! was defined but had not thought deeply about why that might conceptually make sense. The majority of students seemed strictly to have a computational conception of factorials, while a handful of students recognized that factorials may have some combinatorial meaning. Even fewer (just two students, 1 and 8) demonstrated a robust combinatorial conception of factorials. We are not saying that the calculus students *should have* been able to give a combinatorial justification (as they might not have been exposed to a combinatorial context), but these findings suggest that students may have existing conceptions of factorials that they may bring to combinatorial situations.

Responses to the Likert-response Questions

We also get a sense of students' conceptions of factorials by evaluating their Likert-response questions. We do not have space to share all of the responses to these questions, but we highlight a couple of points of discussion. First, the answers to some of the Likert-response questions suggested that many students do associate factorials with counting, if only vaguely, even if this was not demonstrated in their initial characterizations of factorials. To elaborate this point, we consider Statement #5 (*Factorials could be used to solve a counting problem like, "How many possible outcomes are possible if you flip a coin ten times?"*), which speaks to the combinatorial nature of factorial. We would expect the answer to this problem to be a 1 (strongly disagree) because factorials are not appropriate for solving this kind of counting problem (the answer should be 2^{10}, as the number of options does not decrease for each successive stage in the counting process). The average of all responses to this question was 2.77, and the standard deviation was 1.64. Looking more closely at the responses, 8 of the 13 students seemed to recognize that factorials were not appropriate for this counting problem. For example, Student 8 responded with a 2 and said, "Factorials are useful when you have problems that involve, like, uh, situations where your choices, diminish? Like, where you do something, and then the next thing you do you have fewer possible outcomes, um, and that's why they're—yeah, that's why that form is useful." However, 5 of the 13 students agreed or strongly agreed with this statement. In discussing this problem, those who responded with 4's or 5's suggested that they associated factorials with any kind of counting problem. For example, Student 7 responded to the statement with a 5 and said, "I think this has to do with probability, and we would always use factorials in probability. So, uh, I think there is definitely a way to use factorials to solve that." This suggests to us that for these students, factorials are vaguely associated with counting in their minds, but that its combinatorial meaning may not be precisely defined.

Similarly, Statement #7 (*n! is the number of ways to rearrange n objects, even if some of them are identical*) provides further evidence of this phenomenon. We would expect the answer to this statement to be a 1 as well, but the average of the student answers was 2.54 with standard deviation 1.51. More closely examining the students' answers, the majority of students understood a factorial to mean counting arrangements of distinct objects, but there were still four students who agreed or strongly agreed with this statement, again suggesting a vague association with factorials and counting.

Given our previous section that highlights the fact that most students had computational and not combinatorial definitions of factorials, it is not surprising that most students likely did not have a robust understanding of how factorials fit in with solving counting problems. However, the student responses to #5 and #7 suggest to us that students are bringing with them pre-conceived ideas about factorials as they relate to combinatorics that are perhaps not clearly or well defined. These findings suggest that instructors of discrete math and combinatorics should be aware of the kinds of pre-existing notions students might have about factorials.

We also saw some evidence of students' ability to see the multiplication in factorials as a counting process, which suggests perhaps a connection between computational and combinatorial conceptions of factorials. A final item to discuss is Statement #13 ($2\times4\times3\times1$ *is the same as 4!*). Every student responded to this statement with a 5, except for Student 17 who gave it a 4 and Student 1 who gave it a 3. The students overwhelmingly justified their responses by appealing to the commutativity of multiplication of real numbers. This underscores this idea that the multiplication in factorials is just the operation of multiplication, which is commutative. So, we want to emphasize that in some sense, the students are correct that, as a numerical result, the product $2\times4\times3\times1$ is the same as the product $4\times3\times2\times1$ – it must be, because the operation of multiplication of real numbers is commutative. However, we would argue that combinatorially, these two products are not "the same." To see this, we must consider the counting processes and how those processes might be structuring the set of outcomes. In terms of Lockwood's (2013) model, the expression $4\times3\times2\times1$ suggests a counting process with four stages, in which the first stage has 4 options, the second stage 3 options, the third stage 2 options, and the last stage 1 option. The expression $2\times4\times3\times1$ suggests something else entirely – that in a four-stage process the first stage has 2 options, the second 4, the third 3, and the fourth 1. There are plenty of counting processes that reflect this idea (for instance, forming outfits from 2 shirts, 4 pants, 3 hats, and a belt), but they are different than the processes that underlie 4!. In addition, while the multiplication $4\times3\times2\times1$ corresponds naturally to a particular organization of the arrangements of four distinct objects, it is less natural to find an organization of those arrangements corresponding to the multiplication $2\times4\times3\times1$. In this way, the orders of multiplication suggest different relationships between the Formulas/Expressions and Set of Outcomes components of the model.

Student 1 is the only student to have addressed this issue, and not surprisingly he is the student who seemed to have the most robust understanding of factorial in the entire study. Student 1 said about Statement #13, "I know they have the same value, but I don't think this one contains information that 4 factorial has." When asked about the information that 4 factorial has, he said, "4 factorial tells us I'm counting something. But, just, $2\times4\times3\times1$also tells us we're counting something, but, um, I don't know." To us, this suggested that 4 factorial had some different meaning for him in terms of counting objects than $2\times4\times3\times1$. When asked if the two had different meanings to him, he said, "Yeah, I feel differently. Like, if you have 2 options for the first place, then 4 options for second, then 3 and 1, you'll have this number of possibilities. But, for 4 factorial, it means you're doing a specific kind of counting, like, hmm, like ordering—yeah, ordering things. Not just counting the number of [possibilities]. I think there's something more." In saying this, he demonstrated a strong understanding of the multiplication principle, and in particular the way in which the order of multiplication corresponds to distinct, temporal stages in the counting process.

In contrast, Student 17, who gave Statement #13 a 4, only justified his response by saying, "It's the, it's commutative, so, it—they mean the same thing." When asked why he put a 4 and not a 5, he answered "Uh, kinda because it's, like, different to put it that way. It's not, it's not what you would normally put as 4 factorial. I mean, I put a 4 because it's not wrong." This suggests that even Student 17 did not think the expressions were different because they have inherently different combinatorial meaning, merely that $2\times4\times3\times1$ is an unconventional but equivalent way of writing 4 factorial. The responses to Statement #13 provide for us an interesting insight about how students might view factorials, and it suggests that there is more to be investigated about how students understand the multiplication within factorials, especially as it relates to counting processes (and how those processes might generate and structure sets of outcomes).

To summarize our results, the students' initial conceptions revealed a couple of salient points of discussion. First, 15 of the students had seen factorials before, even though only 6 of them had taken coursework beyond discrete mathematics. Their initial definitions revealed that students predominantly conceptualize factorials computationally, although, while only two initially gave a combinatorial definition of factorial, four additional students suggested that they understood that factorials were related to counting (even if imprecisely). The students' responses to the Likert-scale questions gave evidence that many of them thought that factorials were related to counting in general (even if imprecisely), and in addition, the Likert responses revealed the variety of ways in which students conceptualize aspects of factorials. Getting a better sense of students' pre-existing conceptions of factorial (particularly the distinction between computational and combinatorial conceptions) is important as we consider what it might mean for students to have a robust understanding of factorial (and how we might help them adopt such an understanding).

Conclusions, Implications, and Directions for Future Research

We have evidence that students without combinatorial experience (or without a clear combinatorial conception of factorial) are coming into counting situations with some knowledge of factorial. It is important for researchers and teachers of combinatorics to have a sense of what kinds of preconceived ideas their students might have about factorials, and highlighting the combinatorial/computational distinction illuminates different conceptions that student may have as they are introduced to factorials in a discrete mathematics or probability class.

Teachers who might introduce the computational definition of factorial (such as in a calculus class or an induction proof) should be aware of the fact that students might eventually need to understand factorials in a combinatorial context. Factorial should be framed not as some meaningless computational fact or rule, but rather as a flexible and efficient way of writing a product. Teachers who teach discrete mathematics should bear in mind that students might have been introduced previously and might have a purely computational understanding of factorial. It may also be beneficial to realize that students' combinatorial understanding of factorial may be reflected in the three components of Lockwood's (2013) model. The formula should relate to counting processes, which should also relate to the sets of outcomes, and computational facility should be explicitly tied to what that might afford combinatorially. Framing factorials in this way can help to develop in students robust and flexible understandings of factorial. Because factorials appears so frequently and are a key aspect of so many topics and formulas in counting problems, it is important for students to understand them well.

There are a couple of potential directions for future research based on this study. Given that our findings are based on a relatively small number of students, it would be possible to investigate conceptions about factorials among a larger set of students. In so doing, we could investigate whether the distinction between computational and combinatorial conceptions of factorials exists more broadly for other students. In addition, the multiplication principle is a key underpinning idea in factorials, more work can and should be done to examine this fundamental idea and even the role it may play in students' reasoning about factorials.

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