Pre-service teachers' meanings of area

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An exploratory study was conducted of pre-service teachers' understanding of area at a public university in the western United States. Forty-three pre-service teachers took part in the study. Their definitions of area and their responses to area-units tasks were recorded throughout the semester. We found a wide gap between pre-service teachers' meaning of area and their use of area-units. Initially, pre-service teachers had weak definitions of area. Over the semester, these definitions were refined, but misconceptions about area and area-units were illuminated in activities involving non-standard units and areas of irregular regions. We conclude that, despite detailed models of children's understanding of area, much work is needed to understand the learning trajectories of pre-service teachers, particularly when misconceptions exist.

Keywords: Area, Geometry, Pre-service teachers, Units

It has been demonstrated time and time again that many current and future elementary teachers have substantive weaknesses in their geometric content knowledge (e.g., Browning, Edson, Kimani, & Aslan-Tutak, 2014). It is notable that very few of the peer-reviewed studies on pre-service teachers' (PTs') geometric knowledge listed here deal specifically with area. Of the 112 studies published on elementary PTs' content knowledge reviewed for the special edition of the Mathematics Enthusiast in which Browning et al's article appeared, only 4 deal with the status of PT knowledge of area (Enochs and Gabel, 1984; Baturo and Nason, 1996; Reinke, 1997, and Menon, 1998). The findings of all four of these articles are similar, each indicating that the PTs under study demonstrated "incorrect, incomplete, and unconnected" knowledge that was very "rule driven" (Browning et al, 2014, p. 344). Perhaps as a byproduct of this issue, Enochs and Gabel (1984) found that a large percentage of PTs were unable to distinguish volume from surface area, a sentiment echoed by Baturo and Nason (1996) as well as Reinke (1997) which both found that PTs tended to conflate methods of finding perimeter with methods of finding area.

We were teaching a geometry course for elementary teachers when we observed that our PTs did not show a consistent understanding of area[1]. As in the literature cited above, our students confused the attribute of area with its measurement when they defined area as "length times width", as well as confusing area with perimeter and volume. At our two universities, PTs had completed a course in arithmetic before enrolling in the geometry course. This arithmetic course heavily emphasizes the meaning of the multiplication operation so our PTs often gave well-developed explanations for why we multiply to find the number of squares in an array. Our instruction, therefore, aimed to emphasize the meaning of area and to separate this from the process of measuring area and from the formula for the area of a rectangle. We did this by taking a more general approach using non-standard units and looking at the area of irregular shapes. In this context, we observed that most misconceptions that our PTs had about area showed up when they engaged in tasks involving non-standard units and conversion of units, tasks that did not involve computations of area using formulae. In class and in this paper, we take the following definitions. *Area* is the amount of two-dimensional space taken up by a 2D shape. An *area-unit*

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[1] All three taught the same course with the same textbook and supplementary materials.

is any two-dimensional object used to measure a 2D shape. Finally, we describe *measurement* as a comparison of the area-unit with the 2D shape that is accomplished by covering the shape with an iteration of the area-units. We began to look for ways to trace the progressions of individual understanding of area over the course of the semester.

In this initial study, our goal was to examine changes in PTs' understandings of area over the one semester geometry course. More precisely, we asked the following questions: As seen in their written work, what area definitions did PTs bring to this course and how did those definitions change over the semester? What ideas about standard and non-standard area units did PTs demonstrate in their written work?

Theoretical Framework

The theoretical background for this study is drawn from the constructivist theory. From this point of view, one generates ideas by fitting new situations into existing ideas. If the situation does not fit, or if it cannot be explained, one modifies one's existing framework or generates new ideas (von Glasersfeld, 1995). From a social constructivist perspective (e.g., Cobb, Wood, Yackel, & McNeal, 1992), PTs' understandings of classroom conversations and actions are interpreted against a background of prior beliefs about the culture of school mathematics – about the norms and expectations for mathematical behavior and thinking in school – as well as against their prior understandings of mathematical ideas. In order to help them work through misconceptions, our instruction put PTs in situations where these prior understandings would be challenged. For example, we asked PTs to find the area of irregular shapes using non-standard units and we constantly required explanations for any answer given. We did this in part because asking about area of rectangular regions yielded responses that could appear correct, when a second look actually showed misconceptions. Our pedagogy is thus very similar to that described in Simon & Blume (1994), though our course used a textbook as an external resource.

Methodology

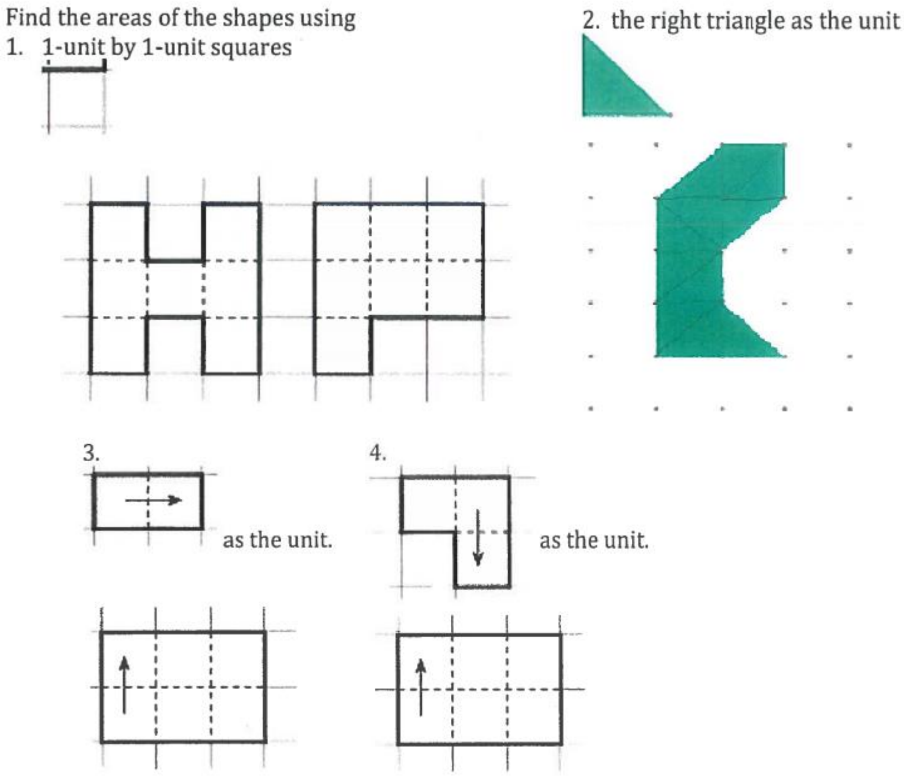
After many conversations together, one of the authors decided to collect data from her classes at a public university. Study participants consisted of 44 PTs from two sections of the mathematics course. One PT was absent for most of the tasks, hence we did not use that data. The PTs were in their third year of undergraduate study. The geometry course is the second course in a two-semester mathematics course sequence for elementary PTs. All participants completed the first course prior to this study. The mathematics textbook for this course is Beckmann (2013), which aligns with the standards of the Common Core State Standards Initiative (2010). The class set-up and the textbook both used an inquiry-based approach (Bruner, 1961) towards learning, where students are encouraged to explore content on their own and discuss with their peers.

The study was conducted throughout the semester and area problems were collected from PTs' in-class writing assignments, quizzes, tests, and from the final exam. The in-class writing assignments (Figure 1) contained questions related to area and area-units and they were repeated multiple times throughout the semester. Each time the answers were discussed in class after PTs got their writing assignment back.

Our choice of tasks here represents a first pass at making a deeper examination of our PTs' area concepts. We intend to take a more precise look at their understandings of area in a future study by adding clinical interviews (Clement, 2000) including conversations about their written work.

Figure 1. Tasks used in the study.

Area Definition Task:
Discuss area of a shape. Give an example.
Units Task 1:



Units Task 2:
What do you understand by 12 m^2?

Data Analysis

Area Definition Task: We analyzed the area definitions following an open and axial coding method (Strauss & Corbin, 1998). Each of the three authors read the PTs' written definitions of area and created a rubric to assign a score to each PT. Then we discussed our rubrics and created a common rubric (Table 1) for assigning scores to each PT's area definition.

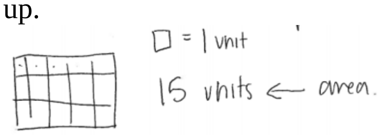
Units Tasks: We analyzed Units Task 1 (non-standard units) and Units Task 2 (*What do you understand by 12m^2?*) by recording each PT's answers. We created a spreadsheet of the PTs' responses to each task so that we could trace an individual PT's progress across the tasks and simultaneously compare responses of all PTs to the same task at the same point in time. Responses to both units tasks were recorded as "correct" and "incorrect".

Results and Discussion

After comparing PTs' definitions of area, their use of non-standard units, and their responses to the question *What do you understand by 12m^2?*, we concluded that their understandings of area differed across these three contexts. Focusing first on responses to the Area Definition Task, we found about 86% of the 43 PTs started with a low understanding of area as measured by scores less than or equal to 3. By our rubric, this suggests that a majority of the study participants did not have a comprehensive understanding of area in the beginning of the semester because their definitions included only “measures space” or “the amount of space that an object takes up”, but did not specify two-dimensional space and made no reference to use of units.

Table 1

Rubric for assigning scores to PTs’ Area Definitions

Score	Description	Corresponding Examples
5	Used covering OR fitting concept, explicitly mentioned measuring a 2D shape AND clearly described a unit of area.	“The area of the shape is the 2 dimensional measurement of the amount of space it takes up. 
4	Used covering OR fitting concept and indicated measuring a 2D shape (either expressed in words or pictures) OR a unit of measurement has been used specifying it as a length or area unit.	“Area of a shape is how much space it takes up in specified units in a 2-dimensional plane.” 
3	Used covering OR fitting concept OR mentioned measuring a shape/ space or outside of a shape (2D is not explicit through words or pictures) OR used *length times width* as an example beside their definition.	“If you were to put something inside it. The area is how much you could fit in.”
2	Discussed area with *length times width* as a requisite part of the definition (not just as an example). No indication of measuring 2D space.	“Length times width, because you want to find the area you have to multiply all of the sides together.”
1	Used volume formulae OR 3D figures as parts of definition OR unclear vocabulary OR did not write anything.	“The area of the shape are the dimensions inside of the shape or the volume of the shape.”

Those scoring 2 described area *only as* “length times width” and those scoring 1 wrote irrelevant or unclear statements with no reference to space at all. Throughout the semester, the same questions were asked and discussed multiple times. On the final exam, a similar question asked for a definition of area compared to perimeter or volume. About 81% of the 43 PTs scored at the level of 4 or 5 on this question. This suggests that PTs’ area definitions improved over time.

Results of Units Task 1 that required PTs to describe the area of a shape using standard and non-standard units showed that only 5 PTs initially identified correct units. Most PTs initially designated all non-square units as “units squared” or as “square units” (see examples in Table 2). Even after three repetitions of this task, each followed by discussions of the answers, only 22 PTs (about half of the total number of PTs) identified correct non-standard units. Analysis of the Area Definition Task suggested improvement in PTs’ understanding of area, but Units Task 1 suggested half of the class still had misconceptions about area. Combining our analyses of Units Tasks 1 and 2, we found that PTs at different levels of area definition answered the two tasks differently (see Table 3). Although 21 PTs reached a level 5-area definition, only 5 of them correctly responded to both units tasks at the end of the semester.

Table 2

PTs' initial responses to Units Task 1

Units in Units Task 1	PTs referring to the corresponding units
1. Square unit	Units^2, Units squared, Unit squares, 1 by 1 squares, Squares, Square units, squares^2, No units, Units, Unit squares^2, 1 by 1 unit squares, 1-unit by 1-unit squares
2. Right Triangle unit	Units^2, Units^2(of the triangles), Units squared, Right triangles, Right triangle units^2, Units^2 triangles, Square units, Triangle units, Units, Right triangles^2, Unit squares, No units
3. Two square unit	Units^2, Units squared, Two square units, Units, Square units, Units of two squares, No units
4. L-shape unit	Units^2, Units, Square units, Units of two squares, No units, L-shape units

This data suggests that, although the idea of "area as covering/ fitting" is mathematically linked to the units used to measure area, these concepts were conceptually distinct for many of our PTs. None of the PTs giving a level 3 definition of area were able to give a correct response to the non-standard units task (Units Task 1), but 3 out of 5 were able to correctly answer the question, "What do you understand by 12m^2?" There were 11 students across all levels of definition who could give good or even excellent definitions of area and explanations of the meaning of 12m^2, but could not apply these ideas correctly in situations involving non-standard area-units. It seems likely that these students had memorized these two ideas, but did not really understand the meaning of area when measured with a non-standard unit.

Table 3

Summary of results of area definition and the two units tasks at the end of the course

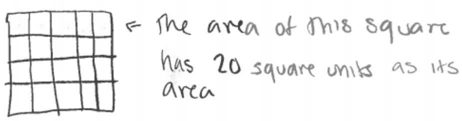
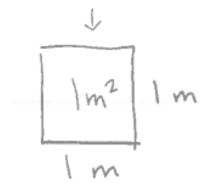
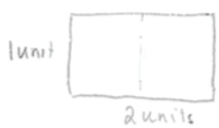
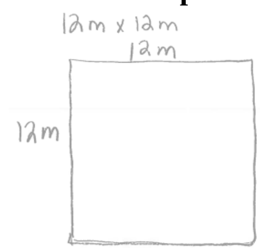
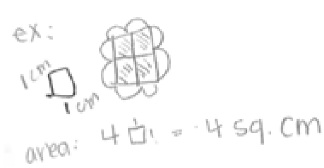
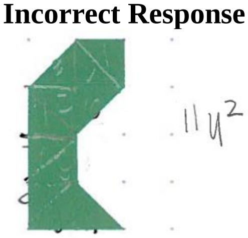
Area Definition Levels	Both Units Tasks Correct	Correct Units Task 1 + Incorrect Units Task 2	Incorrect Units Task 1+ Correct Units Task 2	Both Units Tasks Incorrect	Total
Level 5	5	6	6	4	21
Level 4	3	5	2	4	14
Level 3	0	0	3	2	5
Below 3	0	0	0	3	3
*Note: The numerical values above denote the number of PTs in each category.					

We can see that these ideas are distinct by isolating individual PT's work by level of area definition. Table 4 shows examples of four individuals' work across the three different tasks in their final attempt. Each row shows a different PT's work. For simplicity, we show only examples at definition level 5.

Conclusions and Implications

Our data clearly shows the ability to write a clear and complete definition of area, including reference to the units used to measure it, does not imply full understanding of area. Although area and area-units are tied together mathematically, these ideas were split in the minds of many of our PTs. This is consistent with a long-standing body of literature illustrating the psychological phenomenon of context-dependent understanding (e.g., Carraher, Carraher, & Schliemann, 1985). The only students who had both units tasks correct were those who had attained a level 4 or 5 definition by the end of the semester. In contrast, even after multiple repetitions, 5 of the 8 PTs (see Table 3) who ended the course still writing definitions at level 3 or below were still not able to correctly complete <u>either</u> of the units tasks. This suggests a well-articulated area definition is a necessary, but not sufficient, indicator of PTs' understanding of area.

Table 4
Four PTs' work on area and units tasks.

PTs	Area Definition Task	Units Task 1	Units Task 2
PT 1	**Level 5** The area of a shape is how many two dimensional units fit inside a flat shape. For example 	**Correct Response** 11 triangular units 	**Correct Response** That an area or space is filled up with 12 1m x 1m units 
PT 2	**Level 5** The space a plane shape takes up on a plane. The area, the space the shape takes up, can be defined as the units that makes up the shape. In this case, the shape is made up of 2 square units. 	**Correct Response** 	**Incorrect Response** 
PT 3	**Level 5** The area of a shape is a two dimensional measurement of space such as how many 1 cm by 1 cm squares can fit in a shape.	**Incorrect Response** 	**Correct Response** 12 1meter by 1 meter squares.
PT 4	**Level 5** Area of a shape is the amount of space the shape takes up. It is two dimensional. 	**Incorrect Response** 	**Incorrect Response** It is 12 square meters. So there are 12 square meters in the shape. It is <u>NOT</u> meters squared.

Just as we need caution when assuming that a correct definition implies understanding of area, we need to consider whether incorrect labeling of area-units (e.g., "12 triangle units^2")

necessarily implies deficient understanding of the units themselves. For example, PTs' experiences of units in science classes (where units are cancelled as if they were variables) might have helped to change their interpretation of "square units" from a correct understanding of this to the incorrect "squared units" or "units squared". The student errors we found suggest that PTs are multiplying words (*inches times inches),* like variables, without regard to the units or the meaning of multiplication.

This study shows there is a need for development of progressions of PTs' understanding on geometric topics. Our results overlap with much of the work done with children by Battista (Battista, 2012; Battista et al., 1998). Using teaching experiments (Steffe, 1983), this research breaks the concepts of area and volume into "levels of sophistication" through which children must pass on their way to full understanding of area and volume. Battista (2012) classified reasoning about area into 8 broad levels, with the first four levels all explicitly about units. This suggests that a very deep understanding of units is required in order to attain a comprehensive understanding of area. At the lowest level described, the child "uses numbers in ways unconnected to appropriate area-unit iteration" (p. 112). At the next lowest level, the child "incorrectly iterates area-units" (p. 112). In contrast, our data from PTs show two different levels that indicated no understanding of the relationship between area and the area-units. Looking only at our PTs' definitions, they had two ways to be incorrect: 1) At our Level 1, PTs gave definitions having nothing to do with area or its measurement (e.g., they defined volume instead), and 2) At our Level 2, PTs gave incorrect definitions that relied on the formula for the area of a rectangle, a definition possibly derived from memorized school learning. Our PTs' responses were consistent with observations described by Simon and Blume (1994) who wrote that many of their PTs had a

> *rote procedure for finding area given two linear measures (expressed in common units of length). According to this scheme, one multiplies the two numbers and expresses the product in "square units", so that the second word in the area referent is the same as the referent for the linear measures. It is likely that for some of these [PTs], square units do not conjure up an image of a square. (p. 485).*

Looking at the other developmental levels found by Battista (2012), our PTs seem to be consistently at his sixth level – they "understand and use procedures and formulas for determining areas of rectangles" – but many have not reached the next level where they "generalize their understanding of area measurement to non-squares and to area-unit conversions" (p. 113). Where there is often overlap between the levels of understanding that we see in our PTs with the literature on children's developing understandings, many PTs have succeeded in school for years despite having misconceptions about area and area-units. This causes a PT's learning path to deviate from a child's.

It is clear that we must study the learning paths of PTs directly, not simply compare their understandings to the development of children. We suggest several improvements for future studies. Tasks should be designed specifically to focus on PTs' understanding of area-units, and written work should be paired with interviews asking them to explain their thinking. Study should focus on those PTs who have completed an arithmetic course for teachers to examine how they relate area formula for rectangles and two-dimensional area-units and the role played by multiplication. With enriched understanding of how our PTs learn geometric concepts, teacher educators will be better prepared to work with them to unravel misconceptions and strengthen and rebuild PTs' mathematics for teaching.

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