Graphs of inequalities in two variables

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In this study, I analyze how preservice secondary teachers represented and explained graphs of three inequalities—a linear, a circular, and a parabolic—in two variables. I then suggest new ways to explain graphs of inequalities, i.e. some alternatives to the solution test, based on the preservice teachers' thought processes and by incorporating the idea of variation. These alternatives explain graphs of inequalities as collections of rays or curves, which is similar to graphs of functions as collections of points in one variable functions and as collections of curves in two variable functions. I conclude the study by applying the alternatives to the solving of optimization problems and discussing the implications of these alternatives for future practice and research.

Keywords: Inequality, Variation, Graph, Preservice secondary teacher

Introduction

Mathematical inequalities are important in mathematics due to their connections to mathematical equations and their applications to real-life situations. There has however been a general lack of attention from the mathematics education community on inequalities. Furthermore, the vast majority of the studies on inequalities discuss the understandings and difficulties associated with solving algebraic inequalities in one variable (Almog & Ilany, 2012; Schriber & Tsamir, 2012; Verikios & Farmaki, 2010). Rsearch on inequalities in two variables is almost nonexistent.

In regards to graphs of inequalities in two variables, which is the main focus of this study, most secondary and post-secondary algebra textbooks (David et al., 2011; McKeague, 2008) explain graphs of algebraic inequalities through the solution test. For instance, for the graph of an inequality, $y < x+1$, a series of steps are performed: draw the graph of $y = x+1$; select one or more points from one of the two regions divided by the graph of $y = x+1$; and plug in the x and y coordinates of those points to the inequality, $y < x+1$. If the x and y coordinates of the selected points satisfy the inequality, $y < x+1$, the region from which the points were selected is the graph of the inequality. If not, the other region is the graph of the inequality.

Although the solution test might be the simplest way to explain the graph of $y < x+1$, it does not provide a valid justification for why the entire region below the line graph of $y=x+1$ is the graph of the inequality. This lack of justification may potentially impede students' sound development of conceptions about mathematical proof. As shown in several studies, many students and teachers erroneously derive the truth-value of a mathematical sentence from the truth-value(s) of one or more particular cases (Harel & Sowder, 1998). The solution test is not much different from the misconception for proof of students and teachers in that it determines the truth-values of $y < x+1$ for *all* points (x, y) in a region from the truth-values of $y < x+1$ for *some* points (x, y) in the region. As such, there is a need for a better justification for graphs of inequalities.

This study has two parts. In the first part, I analyze interview data in order to answer the research question: "How is preservice secondary teachers' instrumental and relational understanding of inequalities in two variables?" I then propose alternatives to the solution test based on the preservice secondary teachers' understanding, but by incorporating the concept of the variable as in the topics of one variable and two variable functions (see for example, Herscobics & Linchevski, 1994; Weber & Thompson, 2014, for the concept of variable in functions). I conclude with the benefits of the alternatives, including their connections to graphs of one or two variable functions and their application to the understanding and solving of optimization problems.

Framework

The theoretical framework related to this study is relational understanding and instrumental understanding by Skemp (1976). According to Skemp, there are two kinds of understanding under the same name, mathematics. One is instrumental understanding, which is knowing "without reasons," and the other is relational understanding, which is knowing both "what to do and why." With an instrumental understanding, for example, one might find a division of fractions, $(a/b) \div (c/d)$, by flipping c/d and by multiplying tops and bottoms to get at ad/bc, but she may be unable to explain why she flips or multiplies. Whereas, with a relational understanding, one might use the relationship between division and fraction and the equivalence relationship in fractions, and attain $(a/b) \div (c/d) = \frac{a/b}{c/d} = \frac{a/b \times (bd)}{c/d \times (bd)} = \frac{ad}{bc}$ using the relationships.

There are advantages of relational understanding over instrumental understanding: The former is more adaptable to novel situations, can grow like an organic substance, and helps learners to remember and sustain knowledge. It however has its drawbacks, such as the length of time needed to achieve understanding and in some cases, students' difficulties in obtaining such understanding. Accordingly, teachers often need to make reasoned choices between the two understandings. Skemp argues that relational understanding is the only adequate understanding for teachers, yet many teachers are equipped with only instrumental understanding.

For graphs of inequalities, the solution test is commonly used for instrumental understanding rather than for relational understanding, as it gives instructions on what needs to be done in order to draw graphs. The critical ideas of variation and the infinitude of points embedded in the algebraic and geometric representations of inequalities are omitted, or at least not salient. In the following investigation, I show how preservice secondary teachers represented and explained graphs of inequalities. The goal of the investigation was not only to examine their instrumental and relational understandings but also to investigate the ideas and difficulties involved in their explanations. The details follow.

Preservice Secondary Teachers' Understanding of the Graphs of Inequalities

Methodology

This investigation was performed as part of a larger project that studied the big ideas underlying learners' difficulties in making connections among representations. The participants of the project were 15 undergraduate mathematics majors on the secondary teaching track at a small doctoral comprehensive university in the Southeast. The level of participants' mathematics backgrounds varied—four taking a precalculus course, one taking Calculus I, and the other ten taking Calculus II or above. The participants were individually interviewed twice, for about one-and-a-half hours each time, in the form of a semi-structured clinical interview. The interviews were recorded with a video camera and were transcribed. Their written responses were also collected.

The interview items that were relevant to this study were two questions from the first interview:

Q1: (a) Find a solution of an inequality, $x+2y-32<0$.
(b) Represent all the solutions of the inequality above in the Cartesian plane.

Q2: (a) Find a solution of a system of inequalities, $y<x^2+1$ and $x^2+y^2>1$.
(b) Represent all the solutions of the system of inequalities above in the Cartesian plane.

The interview questions were designed and queried to invoke both the instrumental and relational understandings of the participants. In Q1 and Q2, I first asked the participants to find a "solution" of an algebraic inequality and of a system of inequalities, in order to examine their understanding of the meaning of the solutions of inequalities in symbolic forms. To graph an algebraic inequality or a system of algebraic inequalities is in fact to represent their solutions in the Cartesian plane. As such, it was important to examine their understanding of inequalities in both symbolic and graphical forms. In addition, studies show that students have difficulty understanding what a "solution" means in algebraic inequalities in one variable (Becarra, Sisrisaengtaksin, & Walker, 1999; Blanco & Garrote, 2007). As little is known about students' understanding of the meaning of solutions of algebraic inequalities in two variables, such an investigation is worthwhile.

During the interview, I asked the participants why they represented the solutions in certain ways or why the x, y coordinates of all the points in the region satisfied the inequalities. I also asked participants to represent mathematical statements that were in word or algebraic forms geometrically on the Cartesian coordinate plane. Some of those statements were created by themselves, such as "x is less than 32 when y=0," and some were created by me (the interviewer), such as "$y<x^2+1$ when x=0."

For analysis, I first used an open coding strategy (Strauss, 1987) to code the participants' mathematical behaviors and understandings. Most of the codes used in this stage were for the correctness of their work as well as the ideas, strategies, and difficulties shown in their written or oral explanations. The initial coding showed some patterns and similarities in their thinking and work, and hence yielded categories and subcategories that led to some hypotheses. I then performed the second stage of coding: reexamining and revising the prior codes, and at the same time performing an axial coding (Strauss, 1987) to focus on the categories and subcategories from the previous coding, and hence confirming or refuting the hypotheses.

The results below are some of the findings from the analysis above. The results included here pertain to the relational and instrumental understandings of preservice teachers and some of the characteristics of their mathematical behaviors that are relevant to the alternative explanations shown in the next section.

Results

Instrumental and relational understanding. For their instrumental understanding, I examined the correctness of their graphs, as instrumental understanding is essentially knowing what to do without reasons. For their relational understanding, I used the two factors—the idea of variation, at least to some degree, and the infinitude of points in their arguments—, as they are critical components in knowing why of graphs of inequalities, a relational understanding. An explanation using the solution test by testing one or multiple points was not considered to be relational understanding.

For the linear inequality $x+2y-32<0$, 7 out of 15 preservice teachers showed an instrumental understanding by correctly representing the graph as the lower half plane of the line graph of $y=-x/2+16$, with 2 of them showing relational understanding to some extent. For the circular inequality $x^2+y^2>1$, 5 of 15 showed instrumental and partial relational understandings to some extent by correctly representing the graph as the region outside the circle graph of $x^2+y^2=1$ and by providing a reasonable explanation for their graph. For the parabolic inequality $y<x^2+1$, 3 of 15 showed an instrumental understanding by correctly representing the graph as the lower part of the parabola graph of $y=x^2+1$, with 1 showing relational understanding to some extent (see Table 1).

None of the explanations by the preservice teachers fully used the idea of variation, as I show in the next section. It was also noteworthy that for the circular inequality, preservice teachers' graphical image of a circle corresponding to the algebraic form $x^2+y^2=r^2$ helped them to explain their graphs of inequalities

Table 1 Finding a solution in and representing the graph of an inequality

		Graph-Correct	Graph-incorrect
Linear: $x+2y<32$	One-solution (Correct)	6 (3*)	5
	One-solution (Incorrect)	1 (0*)	3
Circular: $x^2+y^2>1$	One-solution (Correct)	5 (5*)	4
	One-solution (Incorrect)	0	6
Parabolic: $y<x^2+1$	One-solution (Correct)	2 (1*)	7
	One-solution (Incorrect)	1	5

* The numbers inside parentheses indicate the number of participants who show relational understanding for graphs of inequalities to some extent.

The following are some examples that showed relational understanding to some extent:

- For the graph of $x+2y-32<0$, "my y-value is never going to be bigger than 16 when $x=0$, so it must be this whole region," and "anywhere other than the line there is going to be either less than or greater than. So if I choose one up here then I am going to have really big x and y so it will make the form greater."
- For the graph of $x^2+y^2>1$, "x square plus y square is equal to 2, which is greater than 1, then that would mean it is a circle slightly larger than the one," and "I guess it is because everything contained in x^2+y^2 with radius of 1. So nothing inside there is going to be greater than 1. It is all going to be less than 1, but everything outside should leave you something greater than 1."

- For the graph of $y<x^2+1$, "this is a parabola going out from here upward. So I am thinking either it is going to be inside the parabola or outside. You can choose negative numbers. Over here it is actually going to be awesome because x is squared but y won't. So y is going to be less than that. So the way I am seeing it is you can have x and y both being negative. So you can have the Quadrant 3."

Knowing the meaning of solutions of an inequality versus representing the solutions graphically. The analysis also showed that knowing the meaning of solutions in algebraic inequalities was not sufficient for successful representations of graphs of inequalities (see Table 1). Although many preservice teachers—11 out of 15 for the linear inequality and 9 out of 15 for the circular and parabola inequalities—understood the meaning of solutions of algebraic inequalities in that they found one or more pairs of x and y values as solutions of inequalities, only about half of those (or fewer for the parabolic case) who provided a solution correctly in an algebraic inequality successfully provided its corresponding graph—6 of 11 for the linear inequality, 5 of 9 for the circular inequality, and 2 of 9 for the parabolic inequality.

There were many factors that contributed to this failure to transfer, from knowing the meaning of solutions in an algebraic inequality to representing the solutions of the inequality as a graph. The two most prominent, which are closely related to the alternative explanations suggested in this paper, were the following:

Lack of understanding that a graph of an algebraic inequality is a visual representation of the solutions of the algebraic inequality. Many preservice teachers who successfully found one or more solutions of the algebraic inequality $x+2y-32<0$ drew a graph of the line equation $x+2y-32=0$, but falsely claimed that the graph of $x+2y-32<0$ was the line, indicating a lack of understanding that the graph of an inequality is a collection of all points whose x and y coordinates satisfy the algebraic inequality. Yet when they were asked to explain why the points on the line were the solutions of the inequality, some were able to reflect on the meaning of the algebraic inequality and corrected their graphs. The following is an example of the characteristic.

Interviewer: So can you give me an example of the solutions?

Student: Well. $x+2y$ is less than 32. You still have a plenty of pairs of numbers that can satisfy that or not satisfy that. I mean I can give you a random x and y. If $y=2$ and $x=1$, that is certainly true.

Interviewer: Ok, now represent all the solutions in the x-y plane. How would you do that?

Student: That is why I was thinking about this. I was trying to solve in terms of x, so I have a number line because it should be anything on the line. Everything on the line should be a solution.

Interviewer: Why are they solutions?

Student: Everything on the line should be a solution. Hold on. If I plug in 0, 16, it is going to be 32. So, it is going to be this, everything below this line (pointing the graph of $x+2y-32=0$). I am trying to think now. Everything above this line, your y is going to be bigger, it is only this point that is not. Well, any point on this line won't work. I am sure there are still other examples to find. If I go below every value of y. Hold on. I have this whole region below. My y-values will never going to be bigger than 16 when x=0.

Lack of ability to represent an inequality when a variable is fixed as a constant. Most preservice teachers did not know how to transfer an algebraic or verbal inequality statement to a geometric object (which was a horizontal or vertical ray) when a variable in the inequality was fixed as a constant. This representational transfer was requested to the preservice teachers who generated such statement by themselves or to some preservice teachers who successfully provided graphs and/or explanations for graphs of inequalities. Out of 5 preservice teachers who were asked to do the representational transfer, only one of them was able to represent her statements as rays. As for the other 4 preservice teachers, they represented "$x+2y+32<0$ when $x=1$," "my y values will never going to be bigger than 16 when $x=0$," "when $x = 31$, y is less than 0," or "when $x=32$, all the solutions of this inequality would be when $y<0$" as the entire region below the line graph of $x+2y+32=0$ instead of as a vertical ray; 2 of those 4 teachers also represented "$y<x^2+1$ when $x = 3$" or "$y<x^2+1$ when $x=0$" as the entire region below the parabola graph of $y=x^2+1$ instead of as a vertical ray.

Graphs of inequalities in two variables: Alternative explanations

As shown in the Results section, some preservice teachers provided somewhat reasonable explanations for graphs of inequalities that included infinitely many points in their arguments. Their explanations however fell short in that they did not use the idea of variation systemically enough to explain their graphs. This study thereby suggests more systematical ways to explain graphs of inequalities in two variables. These alternatives utilize preservice teachers' ideas of fixing a variable by a constant and their use of the graphical image of curves as shown in the circular inequality, yet fill the gaps in their work by incorporating the idea of variation.

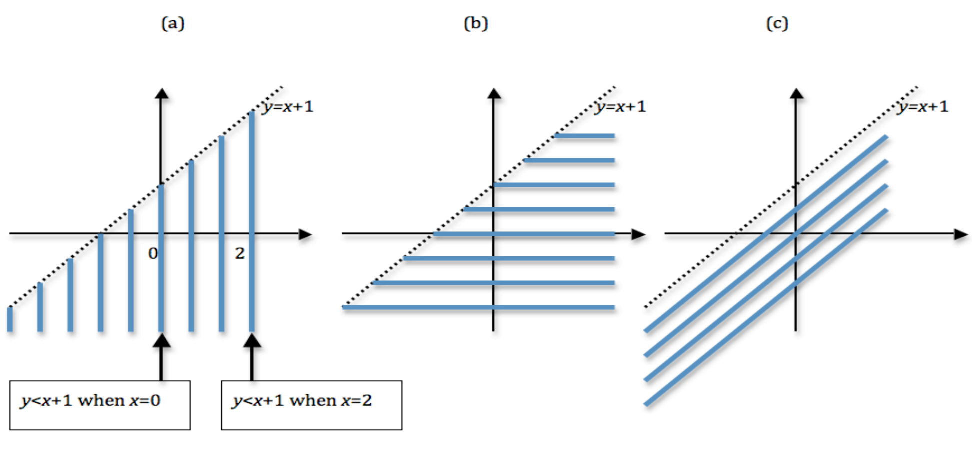

Figure 1 Graph of the inequality, $y < x+1$

Using the inequality $y < x+1$, as an example, the graph of an inequality in two variables can be understood as (a) a collection of vertical rays, $x=c$ and $y < c+1$, if the x variable is kept constant; (b) a collection of horizontal rays, $y=c$ and $c-1 < x$, if the y variable is kept constant; or (c) a collection of lines, $y-x=c$, with $c<1$, if $y-x$ is kept constant. In the first case, shown in Figure 1(a), the graph of $y < x + 1$ when $x=0$ is the collection of points, $(0, y)$, such that $y < 0+1$, which is an open vertical ray on the y-axis. Similarly, the graph of $y<x+1$ when $x=2$ is the collection of points $(2, y)$ such that $y< 2+1$, which is an open vertical ray on the line $x=2$. As x can be any real value, the graph of the inequality is the collection of all those open rays, which forms the entire lower half plane bounded by the line graph of $y=x+1$. The same line of thinking works for the second case, shown in Figure 1(b). Instead of the x variable, the y variable is kept constant; as such, the graph is a collection of open horizontal rays, which forms the entire lower half plane bounded by the line graph of $y=x+1$. In the third case, shown in Figure 1(c), $y < x+1$ is equivalent to $y-x <1$; as such $y-x$ is kept constant with a value less than 1. Whether $c =1/2$, 0, or -1, $y-x =c$ forms a line with the slope 1 and the y-intercept c; as such, the collection of such lines determines the lower half of the line graph of $y=x+1$, as the graph of the inequality $y < x+1$.

Discussion and conclusions

The suggested alternatives are more than merely different explanations for graph of inequalities in two variables. The true benefits of the alternatives are their connections to one and two variable functions as well as their applications to real-life optimization problems, which are one of the most importance uses of inequalities.

To elaborate, the graph of an inequality $y<f(x)$ as a collection of rays is an extended understanding of the graph of $y=f(x)$ as a collection of points, and an understanding that can lead to the graph of $z=f(x,y)$ as a collection of curves. In order to graph the equalities and inequalities, a learner performs an action of fixing a variable as a constant value, $x=c$ for example, and then finds the value of the other variable in the case of $y=f(x)$ or the relationship between the other variables for the case of $z=f(x,y)$. Such an action then yields a geometric object—a point $(c,f(c)$ in a plane in the case of $y=f(x)$; a ray, which is the graphical representation of $\{(c, y)|y<f(x)\}$, in a plane in the case of $y<f(x)$; and a curve $z=f(c,y)$ in a 3-dimensinal space in the case of $z=f(x,y)$. The graph is then a collection of all geometric objects, with the x variable running through all constant values in the domain. In this regard, the above alternatives provide consistency in mathematical thinking related to graphing through the concept of variables in functions, equations and inequalities.

The idea embedded in the alternatives can also help students to solve real-life inequality-related problems, such as those in the *Cookies* unit in the high school mathematics curriculum, *Interactive Mathematics Program*. When finding the maximum profit from the profit function, $f(p, i) = 1.5p + 2i$, with various constraints (represented as a region determined by linear inequalities), students can consider all points on a horizontal (or vertical) line segment by keeping p (or i) constant; they can then understand that the maximum can only occur at the upper boundary points of those segments, which consist of three linear equations. Students can then determine the maximal profit by considering the values of $f(p, i) = 1.5p + 2i$ with constraints given as linear equations, which are relatively easy. This approach not only is different from the strategies in the *Cookies: Teacher's Guide* but also brings a different kind of understanding to the problem. This line of reasoning also aligns with calculus ideas in that the partial derivatives, $f_x(x, y)$ and $f_y(x, y)$, with x or y kept constant, play a critical role in the determination of the extrema of $f(x, y)$.

The alternatives proposed in this study are suggestions based on preservice teachers' understanding of graphs of inequalities and on research on the graphs of one variable and two variable functions. Future research should examine the effects of or problems with implementations.

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