ON SYMBOLS, RECIPROCALS, AND INVERSE FUNCTIONS

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In mathematics the same symbol – superscript (-1) – is used to indicate an inverse of a function and a reciprocal of a rational number. Is there a reason for using the same symbol in both cases? We analyze the responses to this question of prospective secondary school teachers presented in a form of a dialogue between a teacher and a student. The data show that the majority of participants treat the symbol $\square^{-1}$ *as a homonym, that is, the symbol is assigned different and unrelated meanings depending on a context. We exemplify how knowledge of advanced mathematics can guide instructional interaction*

Keywords: Scripting, inverse element, inverse function, reciprocal, homonymy, polysemy

There is an ongoing conversation in mathematics education research on teacher knowledge and its various facets (e.g., Rowland & Ruthven, 2011). One important focus within this discussion is secondary teachers' "advanced mathematical knowledge" (AMK), defined as knowledge acquired during tertiary education (Zazkis & Leikin, 2010). Is this knowledge essential, or even useful, in teaching? Research demonstrated that teachers' opinions on the matter differ considerably, ranging from "irrelevant" to "extremely important" (ibid.)

However, even teachers who claim that AMK is essential for their teaching have difficulty in providing particular examples or recalling teaching scenarios where their AMK was utilized. Our study provides an example where a teacher's knowledge of advanced mathematics can shape an instructional interaction.

The Study

Twenty two prospective secondary school mathematics teachers participated in the study. The participants held degrees in mathematics or science and at the time of data collection were enrolled in a problem-solving course, in the last term of their teacher education program. One of the goals of the course was to draw connections between undergraduate mathematics and school mathematics, and in doing so deepen their knowledge of school mathematics.

During the course the participants had several experiences with script writing assignments – assignments in which they are asked to compose an imagined conversation between a teacher and a student (or students), following a given prompt (e.g., Zazkis, 2014). Our data consists of participants' responses to the Scripting Task, presented below. The task invites participants to write a dialogue for an imaginary interaction between a teacher and a student, related to the appearance of (-1) as a superscript, that is, symbol $\square^{-1}$.

The Scripting Task

The Task is presented in Figure 1. In Part 1 of the Task, the participants were asked to extend the dialogue following the presented prompt. They were explicitly asked to imagine themselves in the teacher's role. In Part 2 the participants were asked to explain their particular choices, which may not be evident from the scripts themselves. While Parts 1 and 2 could be overshadowed by pedagogical considerations, in Part 3 of the assignment we sought the explanation for a "mathematically mature" colleague. This was aimed at liberating the participants from considering the mathematical constraints of the audience and enabling them to project their personal mathematical knowledge. We assumed that participants' personal understanding of the situation and the chosen explanation for students could be different.

Task analysis

In mathematics, a character followed by a subscripted (-1) – such as in f^{-1} and 5^{-1} – represents an inverse element in a group structure. That is, a group element (A^{-1}) is considered to be an inverse of another group element (A), if a binary operation (*) involving the two element results in an identity element (I). Symbolically, this relationship is given as $A * A^{-1} = A^{-1} * A = I$

The two cases presented in the Scripting Task, that is, the cases of f^{-1} and 5^{-1}, differ in terms of the set and the binary operation. For the set of (rational without zero) numbers, the implied binary operation is multiplication, and the multiplicative inverse of 5 is 5^{-1}, as $5 \times 5^{-1} = 5^{-1} \times 5 = 1$, where 1 is a multiplicative identity. For the set of bijective functions, an inverse function satisfies $f \circ f^{-1} = f^{-1} \circ f = id$, where the binary operation here is the composition of functions, and *id(x)=x* is the identity element.

Scripting Task

Part 1: You are given the beginning of an interaction between a teacher and a student and your task is to extend this imaginary interaction in a form of a dialogue between a teacher and a student (or several students). You may also wish to explain the setting, that is, the circumstances in which the particular interaction takes place.

T: So today we will continue our exploration of how to find an inverse function for a given function. Consider for example f(x) = 2x+5 Yes, Dina?

S: So you said yesterday that f^{-1} stands for an inverse function

T: This is correct.

S: But we learned that this power (-1) means 1 over, that is, $5^{-1} = \frac{1}{5}$, right?

T: Right.

S: So is this the same symbol, or what?

Part 2: You are also asked to explain your choice of approach, that is, why did you choose a particular example, what student difficulties do you foresee, why do you find a particular explanation appropriate, etc.

Part 3 (optional): The way you understand the idea yourself could be different from the way you explain it to a student. If this is the case, please indicate how you could clarify the issue for yourself, or for a "mathematically mature" colleague.

Figure 1: Scripting Task

Theoretical Constructs

We rely on the theoretical construct focus of attention (Mason, 2010) and on the linguistic constructs of homonymy and polysemy (Durkin & Shire, 1991). In what follows we briefly introduce each construct and describe how it relates to the work of teachers.

According to Mason (2010) learning involves transformation of attention. In particular, "learning has taken place when people discern details, recognize relationships and perceive properties not previously discerned, through attending in fresh or distinct ways, and when they have fresh possibilities for action from which to choose. Learning necessarily involves shifts in the form as well as the *focus of attention*" (p. 24, our italics). In line with this view, we claim that teachers' work is geared towards focusing students' attention in any given instructional interaction. This choice of focus is intended to draw students' attention to similarities and differences, to stress some aspects of the mathematical concepts and procedures, necessarily ignoring other aspects. For example, in the case of the two appearances of superscript (-1), a teacher may choose to highlight the differences in the procedures implied by the common symbol, or to focus on the unifying idea of 'inverse' element.

Among variously nuanced definitions for homonymy and polysemy, we follow the definitions of Durkin and Shire (1991). According to these authors, homonymy denotes the property of some words to share the same form but point to distinct meanings. Polysemy refers to the property of some words to have different but related meanings, and some shared sense. The context in which the words appear determine their intended meaning.

Durkin and Shire exemplified homonymy and polysemy between the mathematical register and the everyday register. For example, 'volume' is an example of homonymy, as it refers to distinct meanings: intensity of sound or a measure of a 3-dimentional object. 'Continuous' is an example of polysemy, where the everyday meaning of "no breaks" is related to the mathematical definition, such as in "continuous functions". Notably, polysemy may results in learners' misinterpretations of mathematical concepts by assigning the everyday meaning of the related mathematical terms (e.g., Tall & Vinner, 1991, Pimm,1987).

Zazkis (1998) extended the idea of polysemy, relating it to the use of terms *within* the mathematical register (rather than between mathematical and everyday registers). Her example focused on the term 'divisor', that can mean a role of the number in a number sentence (in $12\div4 = ?$ 4 is the divisor and 12 is the dividend ; in $25\div4=?$, 4 is the divisor and 25 is the dividend) or a number-theoretic relationship (4 is a divisor of 12, 4 is not a divisor of 25). Mamolo (2010) further extended the idea of polysemy within the mathematics register to mathematical symbols. She discussed different but related meanings of the '+' symbol denoting a binary operation among elements in different sets.

Of our interest here is another symbol, $\square^{-1}$, its meaning, as determined by different mathematical contexts, and its homonymous or polysemous interpretations by the participants in our study. We posed the following research question: In the two appearances of superscript (-1), what similarities and what differences do teachers identify and focus students' attention?

Data analysis

In analyzing each script (Part 1 of the Task) we identified what is stressed and consequently what is ignored in considering the appearance of superscripts (-1) in the contexts of numbers and functions. In other words, we considered what similarities and what differences the participants identify in the two uses and how they chose to communicate these issues to their students. We confirmed our analysis of the scripts with participants declared pedagogical intentions outlined in their responses to Part 2 of the Task, and identified the intellectual needs of students that the script writers aimed to address. We further attended to Part 3 if it was included in the submission in an attempt to identify whether their choice of pedagogical approach differed from their personal mathematical understanding of the situation.

In the beginning of data analysis, following our mathematical analysis of the Task and some informal conversations with teachers, we considered two extremes:

- A group theory approach, where $\square^{-1}$ stands for the same notion of inverse in a group structure and 5^{-1} and f^{-1} are particular instantiation of inverses in this structure (pointing to different sets of elements and different operation).
- The common symbol $\square^{-1}$ is seen as a homonym, signaling different, context dependent, interpretations for numbers and for functions.

The attempts to classify each response to the Task in terms of association with either these approaches resulted in adding the third abundant possibility, in which similarities beyond the common symbol are sought and exposed through different means. We explained these approaches in terms of *polysemy*, that is, signaling to different but related interpretations of the symbol $\square^{-1}$.

Results and Analysis

In the data we identified the main focus of attention in each script. We first describe the approaches that resulted from considering $\square^{-1}$ as a homonymous symbol. We then turn to scripts in which $\square^{-1}$ was seen as pointing to related meanings, which we describe in terms of polysemy. No script adopted a group-theoretic approach, focusing on the mathematical meaning of "inverse element" with respect to an operation. Surprisingly, there were no noticeable differences between the approaches of students who majored in mathematics and those who majored in science. We exemplify participants' responses via excerpts from their scripts and the accompanying explanations

Focusing on homonymy

Fourteen (out of 22) participants interpreted the appearance of superscript (-1) as a "homonymous symbol", that is, the same symbol applied to different unrelated ideas, the meaning of which is determined by the context. For example, Alan (Excerpt 1) after attending to the terminology, focused on the "neighbor" of the (-1).

Excerpt 1 (Alan): "depending on what it's beside"

Teacher:	It's important to recognize that constants and variables are different from functions. A function takes in a constant or variable and outputs something new based on certain rules. It's like a recipe book. When a function such as $f(x)$ has a power -1 beside the f, it becomes the inverse function. If the power -1 is beside a constant or variable, it means reciprocal.
Student:	So even though it's the same little -1, <u>depending on what it's beside</u>, it can mean either a reciprocal or an inverse?
Teacher:	Exactly.

Alan commented:

"Yes, the two symbols are the same. They both look like exponents, but if you look to what the "exponent" is being applied, it will tell you the meaning of the -1. If the -1 is found above a variable or a constant, then it is understood as an exponent and means a reciprocal. When the -1 is found above a function, it is understood as an inverse function.

$f^{-1}(x)$- *inverse*	5^{-1} - *reciprocal*
$sin^{-1}(x)$ – *inverse*	x^{-1} – *reciprocal*

It is important that students understand the difference between the cases"

Alan's explanation highlights the context in which the symbol appears in order to determine whether it refers to a reciprocal or to an inverse function. For a student, "what is beside" serves an indicator for determining the meaning intended by the mathematical context. In a similar way, highlighting the differences, another participant focused on the letters that determine the "neighbour":

To help students cope with the perceived problematics in dealing with a homonymous symbol, eight participants appealed to analogies of other context-dependent notions. The goal of the analogies appears to convince students that context-dependency is a common phenomenon, which is not unique the superscript (-1). In Rob's script a teacher draws an analogy between an inverse and the word 'set'.

Excerpt 2 (Rob): Analogy to 'set', "context changes its meaning"

Student:	So is this the same symbol, or what?
Teacher:	Yes, it's the same symbol, but it doesn't mean exactly the same thing.
Student:	That doesn't make sense.
Teacher:	Think of it this way: ($^{-1}$) means 'inverse' and your examples are different kinds of inverses. This symbol is used in both contexts, and the <u>context changes its meaning</u>.
Student:	That's so confusing.

Teacher:	I'm going to give you an example. Consider the word 'set.' I could say to you, "Dina, could you 'set' the table?" or I could say "Dina, did you see the sun 'set' last night?" Do you see the difference?
Student:	Yes…
Teacher:	The word 'set' is used in different contexts and those contexts show you which meaning I am using. It's the same with the symbol for 'inverse.'

In this excerpt the teacher acknowledged that in both cases the symbol (-1) points to an inverse. In considering the different meanings of this word, the script-writer builds an analogy with a homonymous word 'set', a word that has different meanings in different contexts. We mentioned above that a majority of participants considered $\square^{-1}$ as a homonymous symbol. Here we see a homonymous word-symbol pair. That is, not only does the symbol get its meaning from the context, but so does the word 'inverse'.
Rob commented:

"By relating the idea of mathematical language to language that the student is more comfortable with, I was able to show the importance of context and the flexibility of the notation we use.

We note that rather than considering the definition of the mathematical term inverse (that is, a binary operation performed on an element and its inverse results in the identity), Rob considers the English language word in its different uses in a mathematical situation. In the next section we examine the linguistic connection further – pointing to the interpretations assigned to the word by considering its synonyms.

Focusing on Polysemy

Seven (out of 22) scripts focused students' attention on a common word, inverse, and the way it is interpreted. (This is in contrast with associating the symbol $\square^{-1}$ with two different words, inverse and reciprocal.) These participants focused on similar features within the two appearances of $\square^{-1}$. To reiterate, the property of a word to point to different but related meanings is referred to as polysemy. As exemplified in Cathy's response, the polysemy is seen in the implied action.

Excerpt 3 (Cathy): Inverse as 'switch'

Student:	So is this the same symbol, or what?
Teacher:	They are the same symbol. Now let's take a step back and investigate this. Dina, can you grab the dictionary at the side of the room for me please and look up the word 'inverse' in a non-mathematical setting
[…]	
Student:	Well, I read that inverse means opposite or reverse, so in a fraction would it mean that we are switching the top and bottom.
Teacher:	Yes, the inverse of a fraction is what we get when we switch the numerator and denominator. Now let's get back to what we are learning about today the inverse of functions. When we are talking functions what are two parts do you think of?
Student:	Left side and right side
Teacher:	Let's look at that what happens to the equation $y = 3x + 4$ if we switch the left side with the right side we get $3x + 4 = y$ what do you notice about these two equations?
Student:	They are exactly the same.
Teacher:	So if switching the left side and right side did not give us what we want, any other suggestions?
Student2:	What about switching the letters x, y?
Teacher:	That is correct; an inverse of the function $y = 3x + 4$ would be $x = 3y + 4$ now let's look at this in more detail.

While the dictionary meaning is assigned to 'inverse' as a noun or an adjective, it is interpreted in case of a fraction as an action of switching between the numerator and denominator. Then the question becomes, what can be "switched" in a function. The first student suggestion – that is considered and consequently rejected – is switching between the left and right sides of an equation. The next suggestion – which is accepted and later examined in further detail is to switch x and y.

Cathy wrote:

> *I choose this method in order to clarify this question to the students because there are a number of terms that are used in multiple ways in math. If we are able to discover what the term actually means then the students may be able to see why the same term is used for both the things identified. I think looking at the definition of inverse and looking at the parts of fractions and equations then the students will hopefully be able to see why the word inverse is used for both."*

Another action, that of "undoing", is featured in the approach presented by Joe, Excerpt 4. In this approach an inverse is associated with an operation that cancels the previous operation and "returns to the starting point". The approach is illustrated in the following excerpt.

Excerpt 4 (Joe): "Get back to the starting point"

Teacher: Let's say I pushed the wrong button on a calculator, and instead of multiplying by 3, I multiplied by 5. Can anyone give me suggestions to what should do next? Should I hit clear and start a long calculation from the beginning?

Student: No! You should just divide by 5.

Teacher: Yes Dina, good instincts. Would dividing by 5 return me to where I was just before I made the mistake?

[The class confirms, with yes and nods.]

Teacher: But what is another way of writing dividing by 5? Yes, Dina.

Student: Putting 5 under 1, so $\frac{1}{5}$?

Teacher: Exactly! So Dina, what do you think it means when I say that the inverse of 5 is $\frac{1}{5}$?

Student: That if I multiplied by five and I want to <u>get back to the starting point</u>, I would multiply by $\frac{1}{5}$ because their multiplication cancels the effects of each other.

[… the dialogue turns to an inverse of a function]

Teacher: So what do you think it means when I say to find the inverse of a function called f?

Student: It means you're trying to find another function related to f, so that it would undo what f did. Return to the starting point.

Teacher: Good! Let's try an example. […]

Joe elaborated on his chosen approach in the following way:

> *"I chose to explain the relations because I felt that trying to convince a student that the power (-1) means two different things was harder than explaining the real reason (that it simply means an inverse). Even though ideas like this might be complex, I think that students should understand that they aren't nonsensical. The people who chose what symbols to use did so for specific reasons. I personally kept the idea of different rules applying to numbers and functions when dealing with inverses. It was not until university Mathematics did I reconcile the two ideas in one overarching idea of an inverse."*

Joe acknowledged that he "kept the idea of different rules" when first introduced to the symbolic notation for a function inverse. However, his study of advanced mathematics helped him adopt a different perspective. Joe's reference to the "overarching idea of an inverse" points to a strong connection he sees in the two contexts of using the symbol.

Summary and Discussion

We described each pedagogical approach in the scripts in terms of the chosen emphasis on the similarities and/or differences in the interpretations of $\square^{-1}$. We attempted to place each approach within the two extremes: common structure of inverses related to different sets and operations, or homonymous symbol referring to unrelated meanings in different contexts.

Participants who considered $\square^{-1}$ as a homonymous symbol amplified the differences by focusing on different context (fractions or functions), different terminology (reciprocal or inverse), and different procedures. Several participants appealed to students familiar experiences and linked different meanings of $\square^{-1}$ to other homonymous words and symbols. Though no participant attempted to connect the script to the mathematical meaning of the term 'inverse', expressing a group-theoretic perspective, several approaches identified polysemy, that is related meanings in the two interpretations of $\square^{-1}$. This was seen by focusing on a common word (inverse) and on a common action implied by the word (swap, undo).

On contextualized knowledge

As mentioned above, the Scripting Task was administered to the participants as part of their work in one of the final courses in their teacher education program. Following the completion of the Task different explanations for the "curious case of superscript (-1)" were discussed in class. Most participants readily recalled or accepted the connection via group theory, referring to the concept of inverse of an element with respect to a particular operation. However, only four participants mentioned the connection in the mathematical meaning of "inverse" in their responses to Part 3 of the Task. (Most participants have not addressed this part or noted that their explanation to students reflected their personal understanding.)

This discrepancy can be explained from the perspective of situated cognition (e.g., Greeno, 1998): Teachers' knowledge is situated within the mathematics classroom and mathematics curricula. In school mathematics there is attention to procedures related to finding an inverse function or multiplicative inverses of fractions. Therefore, the majority of participants have chosen to focus on procedural knowledge expected from students, situating their scripts relative to school curricula, and possibly mimicking how they were taught in school.

Same name to different things

Henri Poincaré is often quoted in saying that "mathematics is the art of giving the same name to different things". He further commented that "It is enough that these things, though differing in matter, should be similar in form, to permit of their being, so to speak, run in the same mold. When language has been well chosen, one is astonished to find that all demonstrations made for a known object apply immediately to many new objects: nothing requires to be changed, not even the terms, since the names have become the same." (Poincaré, 1908)

In English, the multiplicative inverse is usually denoted as 'reciprocal'. Echoing Poincaré, we recognize here "different names for the same thing", which obscures the speakers of English ability to recognize the connection in their different views of $\square^{-1}$. In several languages, where reciprocal for a fraction and inverse for a function are denoted by similar or close words the connection is easier detected.

We agree with Zaslavsky (2009) that "identification of similarities and differences between objects along several dimensions [...] is fundamental mathematical thinking". The Scripting Task discussed in this paper helps highlight similarities and differences in terminology, in structure and in procedures. It provides an impetus for strengthening teachers' personal mathematical knowledge by connecting the ideas of advanced mathematics to a classroom situation.

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