Physics: Bridging the embodied and symbolic worlds of mathematical thinking

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Physics spans understanding in three domains – the Embodied (Real) World, the Formal (Laws) World, and the Symbolic (Math) World. Expert physicists fluidly move among these domains. Deep, conceptual understanding and problem solving thrive in fluency in all three worlds and the facility to make connections among them. However, novice students struggle to embody the symbols or symbolically express the embodiments. The current research focused on how a physics instructor used drawings and models to help his students develop more expert-like thinking and move among the worlds.

Keywords: Embodied and symbolic worlds of mathematical thinking; visualization; physics

Introduction

Mathematics educators have long been fascinated by the power of visualization for learning and teaching mathematics. For example, researchers have cited the theoretical framework in Tall and Vinner's (1981) *Concept image and concept definition* paper over 1600 times. Presmeg (2006) reviewed over 20 years of papers from the *Psychology of Mathematics Education (PME) Proceedings* and found that there is great interest in the topic of visualization. For example, Dreyfus (1991b) stated during his plenary address at PME-15: "It is therefore argued that the status of visualization in mathematics education should and can be upgraded from that of a helpful learning aid to that of a fully recognized tool for learning and proof" (p. 33). Presmeg's review concluded with the statement that: "An ongoing and important theme is the hitherto neglected area of how visualization interacts with the didactics of mathematics. Effective pedagogy that can enhance the use and power of visualization in mathematics education is perhaps the most pressing research concern at this period" (p. 227). Almost two decades later, Presmeg's proposed list of 13 "Big Research Questions" pertaining to the topic of visualization still remains unanswered. Some of her questions include: "How can teachers help learners to make connections between visual and symbolic inscriptions of the same mathematical notions? How may the use of imagery and visual inscriptions facilitate or hinder the reification of processes as mathematics objects? How may visualization be harnessed to promote mathematical abstraction and generalization? What is the structure and what are the components of an overarching theory of visualization for mathematics education?" (Presmeg, 2006, p. 227).

The overarching aim of the current paper is to find out how an expert visualized mathematical ideas and how he subsequently helped his novice students hone their own visualization skills. In this research, we used physics as a case study to investigate how an instructor engaged over 200 students to use visual representations, in particular diagramming physics problems, during lecture and assessments.

Theoretical Framework

We employed Tall's (2013) three-world model of mathematical thinking (conceptual embodiment, operational symbolism, and axiomatic formalism) to describe the possible tensions that novices may face while learning physics. The embodied world involves mental images, perceptions, and thought experiments; the symbolic world involves calculation and algebraic manipulations; the formal world involves mathematical definitions, theories and proofs. In Tall's (2008) view, "all humans go through a long-term development that builds through embodiment and symbolism to formalism" (p. 23). Bridging between the embodied and symbolic worlds is of critical importance according to Tall: “A curriculum that focuses on symbolism and not on related embodiments may limit the vision of the learner who may learn to perform a procedure, even conceive of it as an overall process, but fail to be able to imagine or ‘encapsulate’ the process as an ‘object’ (p. 12).

In the current research, we operationalized the embodied world as demonstrations, real world examples, and models that represent real phenomena. We operationalized the symbolic world as the mathematical operations and computations, such as vectors and calculus, used to solve physics problems. Finally, we considered the formal world to be the rules, laws, and abstract quantities of physics, such as conservation laws, concepts of fields, and energy. In this work we focused on the embodied and symbolic worlds, although using the formal structures of physics was an important course goal held by the physics instructor we studied.

In our perspective, physics must “bridge” the embodied and symbolic worlds of mathematical thought (Figure 1). Expert physicists and engineers embody problems by visualizing them with diagrams, graphs, and schematics prior to solving them symbolically, while novice physics students will “plug and chug.” Students have limited experience relying on visualizations to help them “make sense” of problems, perhaps due to expectations developed from computations in their mathematics classes. We have depicted the bridge that one experienced physics instructor created for his students to move between the embodied and symbolic worlds. The instructor put several connected support pillars in place, including classroom demonstrations of physical phenomena, a student response system that allowed real-time communication with the instructor, and peer instruction. The experienced instructor acted as a guide for his novice students as they traversed unfamiliar territory.

Our current research question investigated how physics bridges the symbolic and embodied worlds of mathematical thought: How does an expert physics instructor construct a bridge between the embodied and symbolic worlds of mathematical thought and help his novice students cross this bridge?

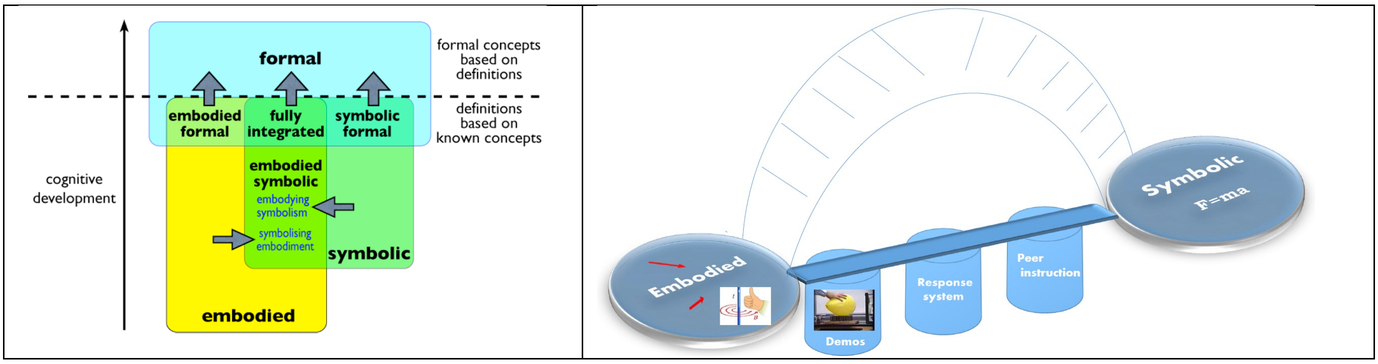


Figure 1. (a) Cognitive development through three worlds of mathematics (Tall, 2008, p. 9). (b) The process of embodying the symbolism and symbolizing the embodiment in physics.

Novice Versus Expert Understanding in Physics

Classic research in cognitive psychology suggests that physics experts and novices approach problems differently (Chi, Feltovich, & Glaser, 1981). For example, experts categorize physics problems on the basis of the underlying physics principle involved, whereas novices categorize the problems on the basis of superficial similarities found across problems. For instance, novices may view all inclined plane problems as equivalent. Experts have a rich, intertwined, hierarchical structure to their knowledge base, whereas novices rely on isolated facts that are not highly structured (Van Heuvelen, 1991). This lack of knowledge structure makes it quite difficult for students to identify the “conceptual unity” in the physics they are taught (Van Heuvelen, 1991). Physics experts effortlessly switch between representations—physical, graphical, schematic, and algebraic—as they reason about problems. It is difficult for novices to learn to identify and use representations that will improve their accuracy and problem solving efficiency (Dreyfus, 1991a; Siegler, 1996).

Van Heuvelen (1991) argues that to get students to think like an expert physicist, students should (1) construct qualitative representations, (2) reason about physical processes through the use of diagrams, (3) construct mathematical representations by referencing the diagrams, and (4) solve the problems using quantitative methods. It is more common for students to attempt a means-end analysis by finding an equation that appears to be appropriate for the problem than it is for them to first employ a sense-making strategy, such as drawing a diagram, to understand how the physical system in question is behaving (Maloney, 2011). Students may resist using diagrams because they do not fully understand the concepts represented in the diagrams, the students have minimal opportunities to develop and practice creating diagrams because they are often passive observers as their instructors create the diagrams, and students' preconceived notions about the way the world works may conflict with what they are being taught in class (Van Heuvelen, 1991). There is educational value in using multiple representations. "The ability to identify and represent the same thing in different representations, and flexibility in moving from one representation to another, allows one to see rich relationships, develop a better conceptual understanding, broaden and deepen one's understanding, and strengthen one's ability to solve problems" (Even, 1998, p. 105). However, visualization is not a trivial task for novice physics students: "We consider the ability to translate between physical and mathematical descriptions of a problem and to meaningfully reflect on or interpret the results as two defining characteristics of a physicist, yet these are areas where our students struggled most" (Wilcox et al., 2013, p. 020119-11).

Sometimes novices who are immersed in a physics course where the instructor values the use of multiple representations to solve problems (e.g., visualization strategies as well as computational strategies) will draw pictures to help them solve problems because that is the course norm. They may not actually understand why they should use the drawings to help them solve problems (Kohl & Finkelstein, 2008). Kohl and Finkelstein (2008) noted that many introductory physics courses do not teach meta-level problem-solving skills, such as highlighting the importance of using diagrams to solve problems. Students may solidify these skills after taking several physics courses, and it is unclear how to effectively teach these skills in an introductory course.

Method

Our qualitative research investigated the ways an expert physics instructor made instructional decisions to help his novice introductory physics students bridge the gap between the embodied and symbolic worlds. The research team consisted of three members: a mathematician, who specializes in mathematics education research (second author), a cognitive psychologist, who investigates learning and transfer of knowledge in the domain of mathematics (first author), and a physicist, Bruce (third author), who focuses on physics education. We examined the daily teaching journals that Bruce kept as he taught a 240-person introductory physics course in the Spring 2014 semester. An innovative aspect of this research is that Bruce is not only a participant in our qualitative research, but he is also an integral member of our research team. Bruce was a consultant throughout the research process from research question design, to data collection and coding, to data analysis and dissemination of results. Bruce was able to confirm whether we had accurately portrayed his teaching techniques and decision-making processes. The research team met weekly to discuss the contents of Bruce's teaching journals and to give Bruce a chance to expand on his weekly lesson plans. Transcripts of these weekly meetings were another source of data.

With Bruce's help, the team qualitatively coded his journals based on the following themes (arranged from most-to-least frequently mentioned): (1) teaching (126 instances, or 35%: goals, real time feedback, question creation, examples, philosophy/best practices, lecture only, sequencing, pedagogical content knowledge), (2) reflections (97 instances, or 27%: on instructor, student understanding/effort, class quality), (3) questions (48 instances, or 13%: instructor questions, peer instruction, in-class quizzes, pre-class questions, formal assessment, qualitative, calculation, homework), (4) visualizations (34 instances, or 9%: student abstract/concrete, instructor abstract/concrete, kinesthetic) (5) students (26 or 7%: real time feedback, questions asked, engagement level), (6) demonstrations (19 instances, or 5%: interactive, illustrative, affective), and (7) mathematics (15 instances, or 4%: qualitative-sense making, quantitative-calculation, use in physics). There were 365 total coded instances. An instance was one or several sentences. Multiple codes could be applied to each instance.

Additional data was drawn from interviews with Bruce in which he created a table of the most difficult concepts that students face in his class. A student in Bruce's course kept a daily journal about his course experiences, and we also investigated answers on an end-of-semester exam and responses submitted during class to Learning Catalytics (http://www.learningcatalytics.com), an online polling system. Students used Learning Catalytics to answer in-class questions, such as multiple choice and free-response questions, to submit drawings for discussion, and to send backchannel questions to Bruce.

The Course Structure and Bruce's Teaching Philosophy

The course that we examined was the second introductory physics course in a two-course sequence focused on thermodynamics, electricity and magnetism, and simple circuits. One of the course goals mentioned on Bruce's syllabus highlighted the importance of bridging the embodied and symbolic worlds: "improve problem solving skills by approaching new problems in a systematic way, plotting out strategies for solution, building and using models, and developing critical thinking skills."

In Bruce's view, the purpose of classes is for students to learn, not for instructors to "teach," and learning is an active process that requires student engagement and effort. Bruce engaged students in active learning activities that included student predictions and explanations. Students prepared for class by completing pre-class reading questions and often worked on peer instruction activities during class (Mazur, 1997).

Bruce's philosophy characterized his beliefs about the connections between math and physics. Physics is about real things that behave in predictable ways, and symbols and numbers are used to represent the properties of these things. If students do not make connections to things, they are doing abstract math, not physics. Students did not need to memorize equations in Bruce's class, but they were expected to understand and use the equations to solve physics problems. His philosophy also characterized the role of models in his physics course. Physics, and all science, is done using models. As approximations to the real world, models allow scientists to consider what is important and ignore what is not. Models can be represented by drawings, graphs, equations, and basic physical concepts. Identifying and applying correct models and representations are important skills to learn.

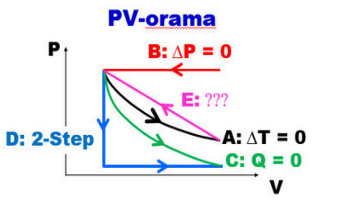
Results and Discussion

Bruce taught this particular course many times, so he could anticipate the concepts that students would struggle with the most. There are many well-known, well-researched stumbling blocks faced by physics students. We asked Bruce to list his lesson plans for all of the difficult course topics that contained some element of visualization. Table 1 provides an example of the use of embodied (e.g., demonstrations, simulations, and visualizations) and symbolic (e.g., visualizations and mathematics) representations by the instructor and students for a challenging concept, using Pressure-Volume (PV) diagrams and models of gas processes to describe real phenomena.

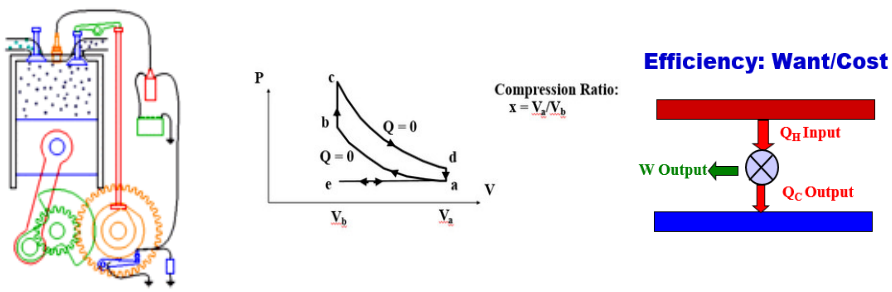
Table 1: Example representations and mathematical tools used for thermodynamics.

Representation	Example	Description	Physics
Demonstration	Heat Engines	Display of Steam and Stirling heat engines. (Class)	Heat exhaust and relative efficiency
	Hard Sphere Model	Styrofoam balls in a cylinder, agitated by a motor. (Class)	Pressure, volume, and energy in an ideal gas.
Simulation	Ideal Gas Model	Simulation of gases with visible bouncing molecules. (Class, Homework)	Pressure, volume, and temperature in gas processes.
	Engine Models	Animated engine illustrations with PV Diagrams. (Class)	Physical applications involving ideal gas processes.
Visualization	PV Diagrams	Draw and interpret pressure vs volume graphs for standard processes. (Class, Homework, Tests)	Sign and magnitude of work, heat, and energy change.
Mathematics	Integration	Work as area under a PV graph. (Class, Homework, Test)	Work and related energies from PV representations.
	Functions	Work, heat, and entropy as logarithms or power laws. (Class, Homework)	Shape and relative slope of curves. Connection with physical properties.

Figure 2 shows visualizations from thermodynamics lessons. The first and last are static images that students create. The visualizations of an operating engine are animations of these static images. Bruce first asked students to make connections between the animated engine and a real engine that was running at the front of the classroom throughout his lesson.



Instructor-provided diagram for in-class questions on the physics of gas processes, asking about energy and temperature.



Three representations of a heat engine showing, from left to right, (1) a schematic real-world operation, (2) details of the gas processes used for quantitative analysis, and (3) a high-level abstraction of energy transfer.

Figure 2: Instructor-created visual representations in thermodynamics.

Students are expected to use these visualizations on assessments, both for group problem solving and on a summative exam. Figure 3 shows student-generated representations of an adiabatic gas process, where $PV^{\gamma} = \text{Constant}$, collected through Learning Catalytics. These give real-time insight into student thinking and use of visual representations. About 60% of students submitted diagrams similar to the first (correct) or second (mostly correct) examples. The next three diagrams illustrate various types of confusion. The final example shows that a few, but not many students attempted to solve the problem, but gave up. Bruce encouraged a fun learning environment. After students submitted their PV Diagrams, some were shown to the class (anonymously) for further discussion by small groups and the entire class to help foster students' use of drawings as they attempted to solve physics problems.

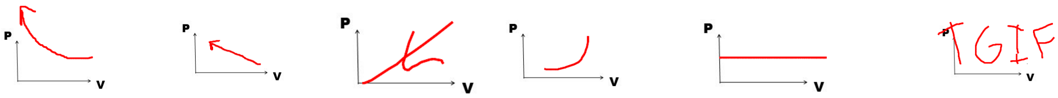


Figure 3: Student submitted in-class drawings for an adiabatic gas process.

In our weekly meetings, Bruce identified that part of students' difficulty with generating the PV diagrams was that many students do not realize that P and V are variables (physics) like X and Y are variables (math): "Let's make P equal to Y and V equal to X, so Y is equal to C over X. Draw that. That one they can do. So this is the ability to generalize that P and V are variables just like you use in math, and therefore you can do the same operations. That's sort of more abstract thinking that a lot of them are working through."

Bruce often broke more complex problems down into smaller, more manageable pieces. "I sort of lead them through it. The first thing I asked them to do is to tell me for this process is the final temperature going to be bigger or smaller? Is the final pressure going to bigger or smaller? It's purely qualitative so that there's no calculation necessary. Then, I would say draw a PV diagram. Then, they actually calculate something. They start out doing it by themselves, and then I have them discuss it. When I'm actually having them calculate things, I have them work together. Then [I'll] step them through the calculation and then sometimes finish up with another qualitative question."

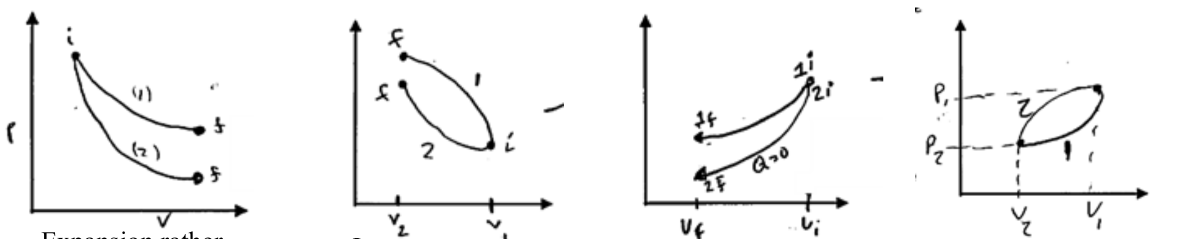
For the visualizations that Bruce showed in class, he allowed the drawing to unfold a little bit at a time: "If it's something that is a picture that I would expect them to create, I try to animate it [on my slides], bring it in piece by piece so that they see how they would build it. So first you draw the axis, you draw this arrow, you draw this arrow, then you label this."

Bruce noted the importance of students creating visualizations on a regular basis: "There are things that I've probably gotten away from a little bit [that] I got back to which is them actually drawing and submitting their drawings. So I'm going to do more of that for the rest of the semester. I kind of got away from that [having them draw pictures and discussing them as a class], and I think that was a mistake. Draw representations that work and then apply those representations to get and organize the equations that will allow you to solve it."

Sometimes students' main obstacle to crossing the embodied-symbolic bridge is simply a lack of mathematical knowledge: "I wish I could guarantee that my students had vector calculus when we were talking about some of this." Sometimes algebra poses a difficulty: "Frankly, we need to get them better at doing algebra. Not really basic algebra, most of these students can do the really simple stuff, but ratios and powers they can't. The music analogy is: algebra is like doing your scales. You know it's just the machinery that you have to be able to do. That should be well practiced, but it's not."

How do students view the experience of crossing the embodied-symbolic bridge? One student agreed to keep a journal about his experiences in Bruce's class. His entries showed the role that diagrams played in facilitating an "a-ha" moment as he bridged the embodied and symbolic worlds. "Often, the diagrams will still be stuck in my head hours later, even if I can't recall the expressions or even much of what was said, and I will keep thinking about them. Then after reading something on Wikipedia, or Googling the material, or reading something in the textbook, the diagram will “resolve” and I suddenly understand why it’s that way. Rarely do I understand this in class when it is taught. However, Bruce’s depictions were absolutely crucial to my understanding of physics. I just could never make sense of it in class. The most frequent “a-ha” moments for me occur during the homework. As I labor to understand the expressions, I’ll recall the diagrams or models and try to understand how the expression describes the model/diagram.”

Students’ self-generated drawings on the final exam (Figure 4) indicated that crossing the bridge is a protracted process: the drawings show gaps in students’ embodied understanding even though their overall exam grades showed that they had a firm grasp on how to symbolically solve related problems. Figure 4 shows a problem that is a comparison of isothermal (logarithmic) and adiabatic (power law) processes. Students did well on this question overall, but some responses still showed student misunderstanding. These examples are from students who received a B on the test, and thus had a reasonable grasp on symbolic solutions to these problems.



Expansion rather than compression

Incorrect curvature and temperatures

Decreasing temperatures

Incorrect temperature and curvature

Figure 4: Student submitted drawings from an exam comparing isothermal and adiabatic gas processes.

Conclusions

This study analyzed teaching diaries and weekly meeting transcripts from an expert physics instructor, who encouraged students to create and submit diagrams through an online response system during his large, introductory physics lectures. Our intention was to begin an investigation of how expert instructors may help novice students navigate the worlds of symbolism and embodiment. Novice students do not possess the dense, interconnected web of physics knowledge that experts have at their disposal (Van Heuvelen, 1991). Students need to be guided or trained as they attempt to cross the bridge between the symbolic and embodied worlds of mathematical thinking. In closing, we offer recommendations that might help physics students make connections between the embodied and symbolic worlds. For example, math instructors may provide students with concrete examples relevant to real world phenomena covered in science and engineering courses. For instance, when instructors are discussing the reciprocal function (Y = 1/X), they could provide an example of the physics concept, Boyle’s law (V = C/P), that states when temperature is constant (C), the volume of a gas (V) is inversely proportional to pressure (P). Physics instructors may need to help students hone their drawing and visualization skills, and this is a skill that should be practiced regularly. Instructors may remind students that their drawings are models, but they are important for identifying the relevant elements of the symbolic and formal worlds for different physical problems. Models are approximations of actual physical systems that can highlight “conceptual unity” and help answer the question “What’s the Physics?” These models help problem solvers to approach a correct answer. Of course, the use of the model must be confirmed through computations and comparison with the real world.

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