Students' formalization of pre-packaged informal arguments

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We gave pairs of students enrolled in a graduate analysis class tasks in which they were provided with a video taped informal argument for why a result held and asked to produce a rigorous proof of this result. This provided a lens into students' formalization process and the various roles these informal arguments played in each pair's proving process. Comparing across several pairs of participants revealed 3 distinct roles informal arguments can play in proving, 1) solely as a starting point, 2) as a reference that can be continually returned to during the proving process, and 3) as a convincing argument that does not inform the proving process.

Key words: [proof, argument, formalization]

Introduction

Acceptable proofs must conform to norms that constrain which assumptions are appropriate starting points, which representations can be used and which inferences are allowed (Stylianides, 2007). These norms vary between mathematical and social contexts (Mamona-Downs, & Downs, 2010). However, regardless of the specific norms at play, it is important to note that the proof generation process need not be constrained by the norms that constrain proof (Boero, 1999). Students may generate an informal argument that provides justification for a particular mathematical result but does not conform to the norms of proof (Harel & Sowder, 1998). A number of researchers have touted the importance of using the formalization of such arguments as a mechanism for proof generation (Gibson, 1998; Raman, 2003; Weber & Alcock, 2004).

Researchers interested in formalization have primarily examined this phenomenon in contexts where it is not specifically prompted for (e.g., Weber & Alcock, 2004, Zazkis, Weber, & Mejia-Ramos, 2015). The major benefit of this approach is that the data are naturally occurring instances of formalization. However, the generation of informal arguments is not particularly common and the successful formalization of these arguments is even rarer. This means that researchers interested in formalization must often start with particularly large data sets that take years to generate in order to explore formalization (e.g., Pedemonte, 2007; Zazkis et al., 2015). Additionally, researchers have little direct control over which specific arguments the students that attempt to formalize start with.

An important under-explored avenue for studying formalization is to use researcher selected pre-packaged informal arguments as part of research tasks (e.g., Zazkis & Villanueva, in press). This gives the researcher the ability to ensure that a much greater percentage of their data is relevant to formalization phenomena and thus eliminates the need for starting with particularly large data sets. Zazkis et al. (in press) presented students with triples which consisted of one informal argument and two correct proofs of a mathematical result, only one of which was a formalization of the informal argument. They observed that a majority of the mathematics majors in their sample struggled to identify which of the two proofs in each triple was the distractor and which was the formalization of the informal argument. They observed that the students' difficulties with making correct assessments could be explained by their focus on a subset of the connections between proofs and informal arguments. This was interpreted as an indication that students had underdeveloped conceptions of what it means for an informal argument to be the basis of a proof.

Methods

In this work we build on Zazkis et al.'s (in press) findings by taking a different approach. We begin by presenting pairs of mathematics students' informal analysis arguments. We then instruct the pair to work cooperatively to turn the given informal argument into a proof. The advantage of this approach is that the entirety of our data set is relevant to formalization. From just 4 pairs of students we are able to generate 8 hours of data relevant to formalization. For comparison Zazkis et al. (2015) started with a data set of 73 students working on seven 15 minute tasks. This generated over 120 hours of data. But only about 8 hours of these data were relevant to formalization.

Participants were recruited from a master's level analysis course. However, some of these participants were undergraduates enrolled in this graduate course and the tasks were accessible at an undergraduate level. The interviews lasted from one to two hours and were conducted in pairs to encourage participants to verbalize their thinking.

The subject area of analysis was chosen because it often has graphical informal arguments that are accessible to students, but the formalization often takes a different form (i.e. epsilon-delta proofs). Two of the proofs used were borrowed from (Zazkis et al., in press). Two additional tasks were identified through informal conversations with faculty who had recently taught analysis.

Pairs of participants were given a statement to prove and then viewed a video clip containing an informal argument that justified why the statement is true. Participants were allowed to watch the video as many times as they needed to until they were confident that they understood the informal argument. Then they were asked to construct a formal proof based on the informal argument. Data was recorded using two video cameras: one recording the participants' gestures and facial expressions and one recording their written work.

Results

Below we discuss three of the four pairs of participants who worked to prove that $\int_{-a}^{a}\sin^3(x)dx = 0$, using the informal graphical argument that the area on each side of the y-axis will add to zero. We illustrate that each pair's approach to formalization, specifically the formalization of the oddness property, differ tremendously in the following vignettes. We use these differences as a launching point for discussing differences in the role informal arguments in general may play in proving.

Pair 1 Vignette: After watching the video, Mark immediately broke up the integral into two pieces, and noted that the two pieces represent the two corresponding areas in the video. They begin to try to manipulate the two integrals by flipping the bounds. They do not mention the oddness property at this point. Chris suggests, "So, as far as creating a rigorous proof, short of integrating sine cubed…" Mark adds, "Yeah, you could just do sine times sine squared and change sine squared to one minus cosine squared, distribute your sine then use u substitution to deal with the sine cosine squared part. That would be one thing."

The pair begin with the integral $\int_{-a}^{a}\sin^3(x)dx = \int_{-a}^{a}(1-\cos^2(x))\sin(x)dx$, and use the u-substitution $u = \cos(x)$ to correctly evaluate the integral. When asked how their solution relates to the informal argument, Chris said, "I would say this was not inspired by the video. If we hadn't watched the video, I think we would have come to the same conclusion in the same manner." Mark points out that their original idea to split the integral into two pieces was inspired by the video.

The interviewer asks them to pursue the two pieces idea in more detail. They show that $\int_0^a \sin^3(x)\,dx = -\cos(a)+1+\frac{\cos^3 a}{3}-1$ using the same calculation as before, and then show that the integral from $-a$ to zero was equal to the additive inverse of this expression by again repeating the method.

Throughout their work on the sin cubed task pair 1 seems to have no conception of the connection between the geometric and algebraic facets of the oddness property. Although they did indicate knowledge of the fact that $\sin(-x)=-\sin(x)$, they never used it in their argument, and it seemed to be a memorized fact unrelated to the graph. They were able to extract the fact the integral from zero to *a* was the additive inverse of the integral from –a to zero, however, the symmetry based reasons for this were not a feature of their discussion. Thus their original proof had nothing to do with the informal argument and their second proof was more inspired by their first proof than the informal argument itself.

<u>Pair 2 Vignette:</u> Heather immediately split up the integral and discussed the impact of the oddness property.

> Heather: I think If we are allowed to know this, split it up into the integral from -*a* to zero and zero to *a* is an idea.
> Rhonda: And maybe use that it's an odd function. Is it odd?
> Heather: Sine cubed odd. Wait, what do you mean by odd?
> Rhonda: They say it's even or odd, I think whenever you put in a negative, it's the same as negative of the regular one. So, like (sketches a graph of x^2) that… I feel like this one is even… because you get the same value.
> Heather: Then when it goes the other way it's odd.

Unlike the first pair discussed earlier, pair two worked to flesh out the graphical meaning of oddness. They flipped the bounds on one of the integrals, then Rhonda said,

> Rhonda: So, I was wondering. Now, since it's all negative here, we know that's equal to negative times what it is with just regular numbers, right? Positive numbers.
> Heather: Wait, what?
> Interviewer: What does it mean for something to be odd for you?
> Rhonda: I think it means whenever you put a negative number into sine cubed, it's the same of negative of that positive number into sine cubed. It's just the negative of the value of the positive (motions with fingers as though picking corresponding points on an odd function).
> Heather: So, if we are given a *k* in whatever, sine cubed of *k* is equal to negative sine cubed of negative *k*, right? Is that what you're going for? And since this is true... We know that this is the sum… the limit of sum of stuff… limit of the sum of stuff of the sine cubed *k* of things. Ok, so how would we do that formally?

Heather then began to talk about Riemann sums and chose corresponding rectangles on either side of zero. Heather explained, "This is nice 'cause it lets us work with the sine cubed value, whereas this (points to integrals) we can't really work with it as much, because it's already inside." Heather and Rhonda were able to talk through this proof construction, though they didn't flesh out the specific details required for a rigorous proof. In working out this proof sketch they commonly went back and forth between working with the graph of sin cubed and working with notation. We interpret this back and forth between the graph of sin cubed and analytic notation to be a back and forth between the informal argument and their proof sketch. Thus unlike pair 1, who almost entirely ignored the informal argument, the informal argument played a continuous role in pair 2's work. It is also worth noting that pair

4 created a similar argument based on Riemann sums, which is not discussed in this paper.

Pair 3 Vignette: Pair 3 began by discussing whether they should directly integrate (like pair 1) or whether they should appeal to the fact that sine cubed is an odd function (like pair 2). Gautam suggested that sine squared can be converted into one minus cosine squared. Cody said, "Yeah, I mean that would show it, but I think it's too much work. We could appeal to the fact that… So, I mean, clearly it's equal to this (splits up integral) and then we appeal to the fact that it is an odd function right here (points to the integrand in the first integral)." They began with the substitution $x = -t$. Gautam and Cody were able to correctly change the bounds and flip the bounds after some discussion. Then they appealed to the oddness property, saying "because *f* is odd, *f* of minus *t* is minus *f* of *t*." They realized that they have missed a negative sign in their substitution, but they were able to find it and write a fully formalized proof.

After finishing this proof, Cody reflected on their proving process: "The oddness definitely made me think of, well, I guess having experience with proofs made me think of $f(x) = -f(-x)$, which is the step you need, along with, so you kind of use the symmetry of the function and the symmetry of the integral. Both of those are a very similar identity because they both involve a flip and a negative sign in some sense. So, you can flip the bounds which introduces a negative sign, and there's another negative sign introduced from substituting, uh, making this substitution."

Pair 3 knew that they needed to use the oddness property in the informal argument. They were able to successfully translate the fact that sine cubed was odd into analytic notation. However, unlike pair 2, they didn't draw any more pictures or give any evidence that they were continuing to think geometrically after they initially extracted the use of oddness and splitting up the integral from the informal argument. So the informal argument provided pair three with a viable starting point, after which they fleshed out the details without further reference to it. In Contrast, pair 2 continually went back to the informal argument during their proving process and pair 1 did not utilize the informal argument during proving.

Discussion

In this study we observed three different patterns in students' usage of pre-packaged informal arguments during proving. It is unknown whether the existence of these patterns remains intact if the mathematical content is changed, or if students tackle the tasks individually. However, since other researchers have noted subject matter influences in other types of formalization studies (e.g., Pedemonte, 2008) we anticipate that these influences on the role of pre-packaged informal arguments are not insignificant. Additionally, how students' usage of pre-packaged informal arguments differ from their usage of self produced informal arguments is an important question for future research.

It bears mentioning that we do not have evidence that any of the three usage patterns of informal arguments provides an advantage over others in relation to producing proofs. These utilizations may simply be different routes to achieve the same goal. We also believe that students are conscious of the role informal arguments play in their own proving. This belief is supported by our participants' reflections on the role informal arguments played in their proving. For example, toward the end of the interview Heather from pair 2 commented on the usefulness of returning to informal arguments:

> Heather: I think sometimes it makes a difference. I think, like, maybe not from this, but sometimes when things are explained informally, it doesn't make sense. So, you play around with it, and after you've played around with it for awhile, then the explanation, if you go back to it, it makes more sense… So, I don't know if, like, watching it at the beginning doesn't help as much as watching it later.

Similarly, Chris from pair 1, after completing his initial proof for task 1, commented about his lack of usage of informal arguments during proving:

> Chris: It's funny, 'cause actually in watching the video. I was like, I don't think that will help [with writing a proof].

Chris and Mark both made similar comments after other tasks in this study. This is an indication that they realized the minimal role that the pre-packaged informal arguments played in their proving process. Further research is needed to explore whether students that treat pre-packaged informal arguments in this way treat their own self-generated informal arguments similarly, or perhaps avoid generating informal arguments altogether.

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Questions for the audience:

1) Is formalizing informal arguments a skill that all mathematics majors should acquire during their undergraduate work, or simply an attribute of some individuals' proving that is not necessarily beneficial for all majors?

2) What are the links between students' ability to generate and use their own informal arguments and their experience with formalizing pre-packaged informal arguments?