A case study of developing self-efficacy in writing proof frameworks

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This case study documents the progression of one non-traditional individual's proof-writing through a semester. We analyzed the videotapes of this individual's one-on-one sessions working through our course notes for an inquiry-based transition-to-proof course. Our theoretical perspective informed our work with this individual and included the view that proof construction is a sequence of (mental, as well as physical) actions. It also included the use of proof frameworks as a means of initiating a written proof. This individual's early reluctance to use proof frameworks, after an initial introduction to them, was documented, as well as her later acceptance of, and proficiency with, them. By the end of the first semester, she had developed considerable facility with both the formal-rhetorical and problem-centered parts of proofs and a sense of self-efficacy.

Key words: Transition-to-proof, Proof Construction, Proof Frameworks, Self-efficacy

This case study concentrates on how one non-traditional mature individual, in one-on-one sessions, progressed from an initial reluctance to use the technique of proof frameworks (Selden, Benkhalti, & Selden, 2014; Selden & Selden, 1995) to a gradual acceptance of, and eventual proficiency with, both writing proof frameworks and completing many entire proofs. This case study further illuminates the well-known, and documented, tendency of students to write proofs from the top-down, and consequently, to be unable to develop complete proofs. (See the case of Willy, who focused too soon on the hypothesis, in Selden, McKee, & Selden, 2010, pp. 209-211). We also consider how this approach to proof construction helped this individual gain a sense of self-efficacy (Bandura, 1994, 1995) with regard to proving.

Theoretical Perspective

In our analysis and in our teaching, we consider proof construction to be a sequence of mental and physical actions, some of which do not appear in the final written proof text. Such a sequence of actions is related to, and extends, what has been called a "possible construction path" of a proof, illustrated in Selden and Selden (2009a). For example, suppose that in a partly completed proof, there is an "or" in the hypothesis of a statement yet to be proved: "If A or B, then C." Here, the situation is having to prove this statement. The interpretation is realizing the need to prove C by cases. The action is constructing two independent sub-proofs; one in which one supposes A and proves C, the other in which one supposes B and proves C.

When several similar situations are followed by similar actions, an *automated link* may be learned between such situations and actions. Subsequently, a situation can be followed by an action, without the need for any conscious processing between the two (Selden, McKee, & Selden, 2010). When students are first learning proof construction, many actions, such as the construction of *proof frameworks* (Selden, Benkhalti, & Selden, 2014; Selden & Selden, 1995), can become automated. A *proof framework* is determined just by the logical structure of the theorem statement and associated definitions. The most common form of a theorem is: "If P, then Q", where P is the hypothesis and Q is the conclusion. In order to construct a proof framework for it, one takes the hypothesis of the theorem, "P", and writes, "Suppose P" to begin the proof. Immediately afterwards, one takes the conclusion of the theorem, "Q", skips towards the bottom of the page, and writes "Therefore Q", leaving enough space for the rest of the proof to emerge in between. This produces the *first level* of the proof framework. At this point, one should focus on the conclusion and "unpack" its meaning. It may happen that the unpacked meaning of Q has the same logical form as the original theorem, that is, a statement with a hypothesis and a conclusion. In that case, one can repeat the above process, providing a *second level* proof framework in the blank space between the first and last lines of the emerging proof. (For some examples, see Selden, Benkhalti, & Selden, 2014).

Prior Research

While studies on students' learning to write proofs have been made before, they have not so specifically focused on proof frameworks. Hazzan (1999) has written about how students cope with the transition to upper level proof-based mathematics, specifically when they take their first undergraduate abstract algebra course. Dahlberg and Housman (1997) were interested in how a student develops his/her concept image when learning a new mathematical concept. They found that students who engaged in example generation and reflection during the study of definitions were able to attain a more comprehensive understanding. Housman and Porter (2003) found a correspondence between students who used transformational proof schemes and those who successfully generated examples when asked to do so. Selden, McKee, and Selden (2010) reported instances of students' tendencies to write proofs from the top down and their reluctance to unpack and use the conclusion to structure their proofs. This study extends that work.

Methodology: Conduct of the Study

We met regularly for individual 75-minute sessions with a mature working professional, Alice, who wanted to learn how to construct proofs. Alice followed the same course notes previously written for an inquiry-based course used with beginning mathematics graduate students who wanted extra practice in writing proofs. The sessions were almost entirely devoted to having Alice attempt to construct proofs in front of us, often thinking aloud, and to giving her feedback and advice on her work. The notes had been designed to provide graduate students with as many different kinds of proving experiences as possible and included the kinds of proofs often found in typical proof-based courses. They covered some sets, functions, real analysis, and algebra, in that order.

Alice had a good undergraduate background in mathematics from some time ago and also had prior teaching experience. She only worked on proofs during the actual times we met. While she usually came twice a week to see us and work on constructing proofs, sometimes when her paid work got a bit overwhelming, she would take a week off. Thus, unlike the graduate students who took the course as a one-semester 3-credit class, Alice worked with us on our course notes for two semesters at her own pace and did not want credit.

We met in a small seminar room with blackboards on three sides, and Alice constructed original proofs at the blackboard, eventually using the middle blackboard almost exclusively for her evolving proofs. After several meetings, she began to use the left board for definitions and the right board for scratch work. She did not seem shy or overly concerned with working at the board in front of us, and from the start, we developed a very collegial working relationship. She seemed to enjoy our interactions as she worked through the course notes. Thus, we gained greater than normal insight into her mode of working. We videotaped every session and took field notes on what Alice wrote on the three boards, along with her interactions with us. For this particular study, we reviewed the first semester videos and field notes several times, looking for signs of progression in Alice's approach to constructing proofs.

The Progression

Our First Meeting with Alice

We introduced Alice to the idea of proof frameworks and explained in detail how and why we use them. We also introduced her to the idea of unpacking the conclusion and mentioned that proofs are not written from the top down by mathematicians. With guidance, she was able to prove "If $A \subseteq B$, then $A \cap C \subseteq B \cap C$." In addition, she worked three exercises on writing proof frameworks--one on elementary number theory and two on set equality. Near the end of this meeting, Alice produced a proof framework for the next theorem in the notes. We felt that she not only understood our reasoning for using proof frameworks, but also how to construct them.

Our Second Meeting with Alice—Her Reluctance to Use Proof Frameworks Surfaces

At the beginning of the second meeting, Alice went to the middle board and produced the same proof framework, as she had done five days earlier at our first meeting (Figure 1).

Theorem: Let A, B, and C be sets. If $A \subseteq B$, then $C - B \subseteq C - A$. Proof: Let A, B, and C be sets. Suppose $A \subseteq B$. Suppose $x \in C - B$. So $x \in C$ and x is not an element of B.	Thus $x \in C$ and x is not in A. Therefore x is in $C - A$. Therefore, $C - B \subseteq C - A$.

Figure 1. The proof framework that Alice produced on the middle blackboard.

Then Alice stopped and after a long silence of 65 seconds, much to our surprise, said, "I have a question for you. I find it very difficult to see the framework. Let me show you how I do it, because somehow I get confused with the framework." We asked her what it was about the framework that was confusing, but she seemingly could not put it into words. So we encouraged her to write the proof the way she preferred. Thus, on the left board, Alice began to write the proof in her own way in top down fashion (Figure 2). She then paused for 15 seconds, and said, "We need to have one more," and wrote into her proof attempt, **"and $x \in A$"** immediately below **"$x \in C - B$"**, indicating with a caret that **"and $x \in A$"** was also part of her supposition (Figure 3). Then, after a 35-second pause, she added to her proof attempt, "**Since $x \in A$ and A is a subset B. Then $x \in B$.**" Shortly thereafter, Alice quietly said, "Oh, a contradiction". This was followed by, "Yeah, 'cause x doesn't belong to B. Yeah, problem here."

Theorem: Let A, B, and C be sets. If $A \subseteq B$, then $C - B \subseteq C - A$. Proof. Let A, B, C be sets. Suppose A is a subset of B. We need to prove that $C - B$ is a subset of $C - A$. Suppose $x \in C - B$. We need to prove that $x \in C - A$.

Figure 2. Alice's attempt at constructing a proof in her own way.

Then, after a ten second pause, Alice said, "The problem is right here, isn't it?" pointing and underlining "***B***" and the statement "**and $\boldsymbol{x \in A}$**." We asked, "And what do you think that problem is?" Alice replied, "I assumed that [pointing to **"and $\boldsymbol{x \in A}$"**], but I do not know. I only know this [pointing to **"*A* is a subset of *B*"**]. We replied, "So that's a good point you've made."

Theorem: Let A, B, and C be sets. If $A \subseteq B$, then $C - B \subseteq C - A$. Proof. Let A, B, C be sets. Suppose A is a subset of $\underline{\boldsymbol{B}}$. We need to prove that $C - B$ is a subset of $C - A$. Suppose $\boldsymbol{x \in C - B}$. We need to prove that $x \in C - A$. $\underline{\textbf{and } \boldsymbol{x \in A}}$**.**

Figure 3. Alice's adjustments to her proof attempt, done in her own way.

After that, for a few minutes, we talked about the structure of proofs, and why we use proof frameworks. Then we asked Alice to elaborate on why **"and $\boldsymbol{x \in A}$"** is a problem. She said, "I didn't write it right. I should have said here [pointing to the blank space to the left of **"and $\boldsymbol{x \in A}$"**] I'm going to make an assumption like **'Suppose *x* belongs to the *A*'**, and then since x belongs to the A and I know that A is a subset of B, **then the *x* will belong to the *B***." She continued, "I also know that x belongs in the $C - B$, because I said it earlier. Then x belongs to the C but ***x* does not belong to the *B***." To which one of us replied, "And then you said something. I thought I heard you say the word **'contradiction'**." Alice explained, "Yeah, I got a contradiction because then I'm saying here [pointing to the board] the x belongs to the B, and the x doesn't belong to the B." We agreed, and she offered, "That assumption [pointing to **"and $\boldsymbol{x \in A}$"**] was bad." We then reiterated why proof frameworks are structured the way they are, and suggested that we could take Alice's original framework (Figure 1) and what Alice had written on the left board (Figure 3), and change the order to write a proof. We proceeded to help Alice do this.

Subsequent Meetings with Alice

As the meetings went on, we observed that Alice became very methodical in her approach to proving, and also somewhat more accustomed to writing proof frameworks. We hypothesize this was because of her technical work experience and perhaps because of her natural tendencies. By the 12$^{\text{th}}$ meeting, Alice had developed the following pattern of working: She would write the statement of the theorem to be proved on the middle board, then look up in the course notes the definitions of terms that occurred in the theorem statement, write them exactly as stated on the left board, and use the right board for scratch work. Indeed, during the 12$^{\text{th}}$ meeting, when she got to the theorem, "Let X, Y, and Z be sets. Let f: $X \to Y$ and g: $Y \to Z$ be 1-1 functions. Then $f \circ g$ is 1-1," she wrote the first and second level frameworks ostensibly on her own, and with some guidance from us, completed the proof and read it over for herself aloud.

By the 19th meeting at the end of the first semester, Alice was more fluent with writing proof frameworks than on the 12th meeting, and she had adopted the technique of writing definitions on the left board and changing the variable names to agree with those used in the theorem statement – all without prompting from us. This is remarkable as our experience has been that many students do not change variable names in definitions even when we suggest doing so, and this can often lead to difficulties. Alice continued meeting with us and working on the course notes at her own pace during the second semester. We plan to continue our analysis of those videos for Alice's continued progression.

Summary of Results

Alice came to us apparently with a reasonable undergraduate mathematics background, some of which she had forgotten. At the first meeting we explained the use of proof frameworks and our rationale for using them, and she practiced producing several of them. However, at the second meeting she told us that she found this way of working confusing. When she attempted her own alternative method of proving, she got into difficulty, and as a result, was more willing to try using proof frameworks again. Over the course of our subsequent meetings that semester, Alice became fluent with writing both first and second level frameworks, and adopted a methodical way of working. As time went on, she was able to complete proofs with less guidance from us. Indeed, she often mainly required some help with the problem-centered parts of proofs. In the following semester, she continued meeting with us and working on the course notes. We feel that, by the end of the second semester, she had developed a sense of self-efficacy (Bandura, 1994, 1995) regarding her proving ability and expect to document that further.

Implications

The initial tendency of many university students to write proofs in a top-down fashion tends to fade after sufficient exposure to writing proof frameworks. One might ask where this tendency comes from. According to Nachlieli and Herbst (2009), it is the norm among U.S. high school geometry teachers to require students, when doing two-column proofs, to follow every statement immediately by a reason. This implies top-down proof construction. However, as noted previously, automating the actions required to write the formal-rhetorical part of a proof can allow students to "get started" writing and exposes the "real problem" to be solved in order to complete the proof (Selden & Selden, 2009b). For this, persistence and self-efficacy are needed.

Discussion Questions

1. What more should we look for when we analyze the second semester videos?

2. In our experience, mathematicians just know how to structure proofs (e.g., including how proofs can begin and end). Apparently, they have tacitly learned this, as well as the importance of "knowing where they are going" (e.g., unpacking the conclusion). How and when do mathematics majors learn this, when not introduced to doing so explicitly via an inquiry-based course like ours?

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