Students' symmetric ability in relation to their use and preference for symmetry heuristics in problem solving

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Advanced mathematical problem solving is marked by efficient and fluid use of multiple solution strategies. Symmetric arguments are apt heuristics and eminently useful in mathematics and science fields. Research suggests that mathematics proficiency is correlated with spatial reasoning. We define symmetric ability as fluency with mentally visualizing, manipulating, and making comparisons among 2D objects under rotation and reflection. We hypothesize that symmetric ability is a distinct sub-ability of spatial reasoning which is more accessible to students due to inherent cultural biases for symmetric balance. Do students with varying levels of symmetric ability use or prefer symmetric arguments in problem solving? How does symmetric ability relate to insight in problem solving? Results from a pilot study indicate that, among undergraduates, there is high variation in symmetric ability. Further, students with higher symmetric ability tend towards more positive attitudes about mathematics. Methods, future research, and implications are discussed.

Key words: [Symmetry, Problem Solving, Heuristics, Cognitive Spatial Reasoning]

Background and Research Questions

Schoenfeld (1987) demonstrates that metacognitively aware problem solvers read, analyze, explore, plan, implement, and verify during their problem solving process. The "analyze" phase involves creating hypotheses of how a problem can be solved. Symmetry is often an easy path in problem solving and broadly appeals to students and mathematicians alike as a major convergence point of mathematics and beauty (Drefus, T., Eisenberg, 1990; Goldin & McClintock, 1980).

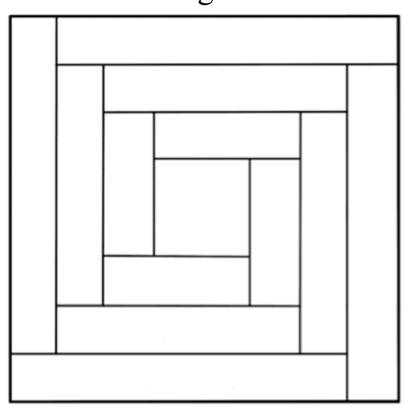
A typical American geometry curriculum is capped around age 16 with construction based proof geometry and trigonometry. Given this typical educational history it is unsurprising that undergraduate students have trouble with geometric transformations, including symmetrical relationships (Rizzo, 2013) a deficit which the CCSSM has addressed by including geometric transformations and symmetry of functions in its content suggestions (Initiative., 2011). Beyond this, however, the utility of symmetry as heuristic has application in multivariate calculus, organic chemistry, applied engineering and design, and physics. Expanding the bounds of a student's ability to use and conceive of multiple solution strategies serves to increase this "analysis" phase of problem-solving. Problem-solving and critical thinking are the main pillars of reformed K-12 curricula (Initiative., 2012; Mathematics., 2000) and are pervasive in PCAST reports (Holdren & Lander, 2012). It is the goal of this study to characterize the relationship between students' ability in and application of symmetry to problem solving. This will serve as a research basis for the continued curricular expansion of treatments of symmetry within the geometric transformations and provide insight into how students currently think about symmetry as a heuristic. Future research might seek to find out how one's symmetric ability can be built upon in challenging problem solving situations.

Previous research with in-service teachers shows that they generally do not use symmetric solution strategies, and are skeptical of the mathematical validity or sufficiency of such solution strategies when working on multiple solution tasks (Leikin, Berman, & Zaslavsky, 2000; Leikin, 2003). Similarly they believe that conventional solution strategies (relying on calculus, algebra, or geometric definitions) are more trustworthy and that they have more confidence in teaching them. This relationship, between ability to think symmetrically, the insightful recognition of when to use symmetric arguments, and preference for/against symmetric arguments among worked out solutions has not been investigated with students. Research indicates that affective/attitudinal factors greatly influence mathematical achievement (E. Fennema & Sherman, 1977) as well as selective processes. Meaning that one's attitudes about mathematics influence one's mathematics performance as well as one's decisions having to do with mathematics. This selection was studied as it related to career choice (Betz & Hackett, 1983). We propose to look at this in relation to problem scale mathematic preferences (ranking of solution strategies). In response to the research background and instructional significance presented here, this research project is designed to answer the following research questions:

1. How do students' attitudes about mathematics relate to their ability with symmetry?
2. How does students' symmetric ability relate to their use and preference for symmetric arguments in problem-solving?

Research Methods

In this section I describe pilot data that has already been collected and describe plans for future data collection. We have developed an instrument to measure students' symmetric ability defined as: a student's ability to mentally visualize, manipulate, and make comparisons among 2D geometric objects and as applied to cultural material in terms of reflectional and rotational symmetry. Our definition mimicks those that Olkun (2003) summarizes of, spatial ability in reasoning, relations, and vizualizations with an added cultural component influenced by research on ethnomathematics (Abas, 2004; D'Ambrosio, 2001; Eglash, Bennett, O'Donnell, Jennings, & Cintorino, 2006). Sample items can be seen in Figures 1-4.

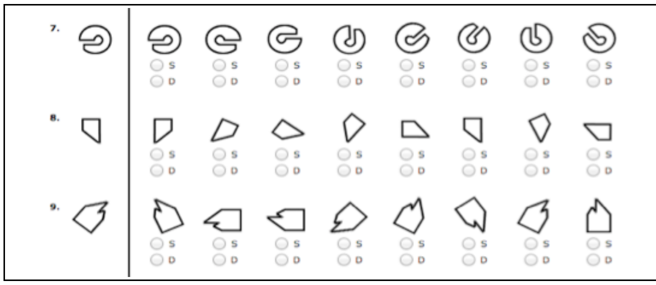


a. How many unique turns of rotation could you perform on this quilt pattern?

b. How many axes of reflection (lines of reflection) does this quilt pattern have? Please draw those that are present.

Figure 1. Cultural item from symmetric ability survey, one of five.

Cultural items were developed by the researchers with consideration of D'Ambrosio's (2001) findings on ethnomathematics: that using local cultural material in curricula, in this case flooring and quilt patterns, symbols and logos, and architecture, increases student engagement. Geometric 2D items were chosen from the literature on spatial reasoning (Ekstrom, French, Harman, & Derman, 1976; French, Ekstrom, & Price, 1963), and attitudinal items were drawn from the Fennema-Sherman confidence, beliefs, and effectance subscales (Fennema & Sherman, 1976). In error, only partial subscales were used in this pilot, future research will use the entirety of the confidence and effectance subscales.



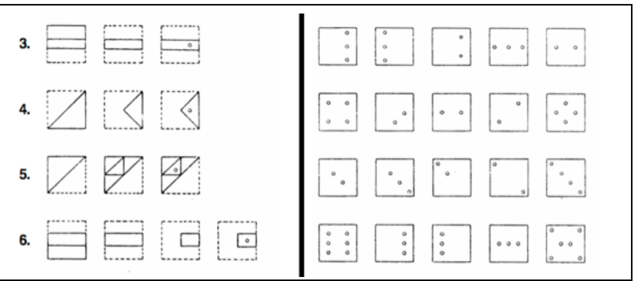
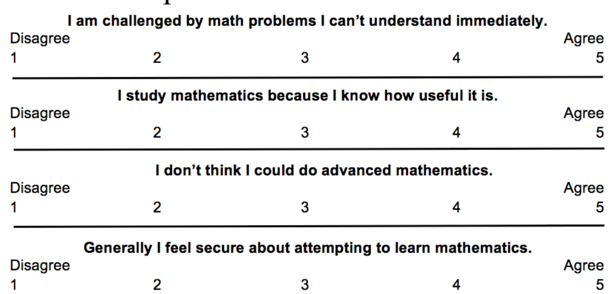


Figure 2. (left) Geometric 2D items from symmetric ability survey, tests rotational ability. Students mark whether the eight images are rotationally symmetric (S) or different (D) from the leftmost image. Three of ten.

Figure 3. (right) Geometric 2D items from symmetric ability survey, tests reflectional ability. Images to the left of the bold line show a series of folds and hole punches done on a square piece of paper. Students choose the image to the right that corresponds to the correct pattern of holes when the paper is unfolded in place. Four of ten.



I am challenged by math problems I can't understand immediately.

Disagree 1 2 3 4 5 Agree

I study mathematics because I know how useful it is.

Disagree 1 2 3 4 5 Agree

I don't think I could do advanced mathematics.

Disagree 1 2 3 4 5 Agree

Generally I feel secure about attempting to learn mathematics.

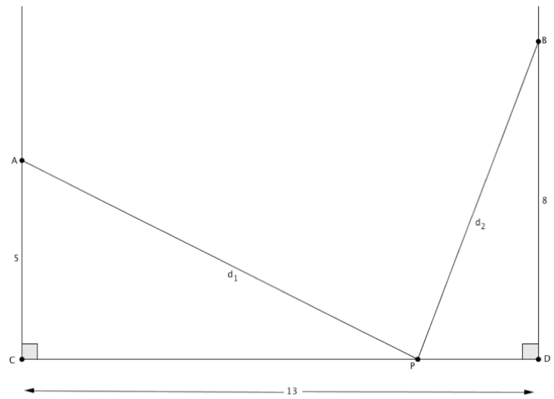
Disagree 1 2 3 4 5 Agree

Figure 4. Attitudinal items from symmetric ability survey. Four of eleven.

Cultural and geometric problems were preceded by instructions with worked examples; students were encouraged to ask for clarification of these instructions as necessary. This timed survey was administered to a small (n=11) pilot sample of students in an introductory general mathematics course at a large northeastern university. Post survey interviews are in process to establish survey validity of the researcher-developed cultural items.

Primary data collection is set for the autumn of 2015. The symmetric ability survey (Fig. 1-4) will be administered to n~100 students enrolled in introductory calculus or more advanced math courses. A subsample, n~20, will be selected to take part in a think-aloud interview centering on multiple solution tasks. We will select students to ensure high variation of symmetric ability and attitude within the subpopulation. Example multiple solution tasks that assess use of symmetric heuristics, open response format, and preference for symmetric heuristic, ranking format, can be seen in figures 5-6. Sample prompts drawn from the cognitive interview protocol can be seen in figure 7. Students must have some access to ideas from calculus to complete preference questions; unfortunately calculus students were not available for the pilot. While students work through two problems of each type, use (Fig. 5) and preference (Fig. 6), prompts like those in Figure 7 will be used to elicit student thinking and understanding. Qualitative and grounded theory methods will be used to analyze audio/video recordings of these problem-solving interviews.

A TV game show contestant must run from her initial starting position A to a long table CD that is covered with chocolate cream pies. The table is 13m long and is 5m away from A. After picking up a pie from the table, the contestant is to race to her partner, who is 8m from the table at B, and give him a faceful of pie as fast as possible. What is the shortest distance in which she can accomplish this feat?



Which solution to the following system of equations is best?

$$\begin{cases} 3x + 2y + z = 30 \\ x + 3y + 2z = 30 \\ 2x + y + 3z = 30 \end{cases}$$

1. To solve a system of three equations with three variables we can solve by pairs of equations. Taking the first and second equations together and the second and third equations together we can remove *z* from both by multiplying out the *z* coefficient and subtracting equations as shown.

$$\begin{array}{cc} 3x + 2y + z = 30 & x + 3y + 2z = 30 \\ x + 3y + 2z = 30 & 2x + y + 3z = 30 \\ & \\ 2(3x + 2y + z) = (30)2 & 3(x + 3y + 2z) = (30)3 \\ 1(x + 3y + 2z) = (30)1 & 2(2x + y + 3z) = (30)2 \\ & \\ 6x + 4y + 2z = 60 & 4x + 2y + 6z = 60 \\ -1x + 3y + 2z = 30 & -3x + 9y + 6z = 90 \\ \hline 5x + y = 60 & x - 7y = -30 \end{array}$$

We now have a system of two equations in two variables. Solving for *x* or *y* in either and substituting will yield an answer of *x* = 5, *y* = 5, and *z* = 5.

2. Notice that the coefficients of x,y,z rotate through the equations: {3,2,1} {1,3,2} {2,1,3}. For all of these permutations to equal 30, *x* = *y* = *z*. The first equation can be rewritten:

$$\begin{aligned} 3x + 2x + x &= 30 \\ 6x &= 30 \end{aligned}$$

Therefore, *x* = 5, *y* = 5, and *z* = 5.

Figure 5. (left) Students may respond to this open question with several solution strategies. Expected solutions include parameterization and minimization, guess and check, and reflection of B about CD to form the straight path AB' and application of the Pythagorean theorem.

Figure 6. (right) Students respond by ranking their preferred solution strategy.

Use:
1. Can you explain how you decided to try solving the problem this way?
2. Can you think of any other ways to solve this problem?
3. How might a mathematician approach this problem?

Preference:
1. Can you rephrase this solution strategy in your own words?
2. How are you defining "best" in this question?
3. Can you think of any other ways to solve this problem?

Figure 7. Sample question prompts from the problem solving interview protocol.

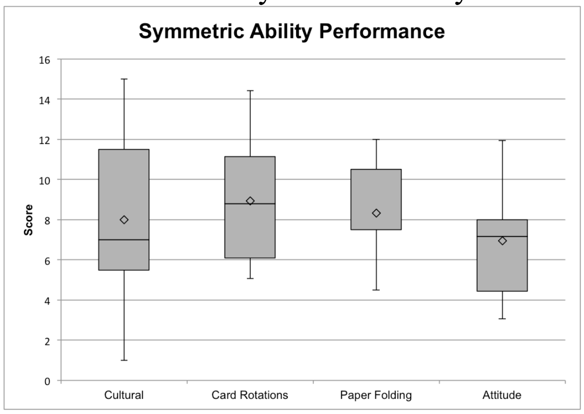
Data Analysis

A rubric was developed to establish this survey as a quantitative measure of symmetric ability and to provide insight into any relationships between cultural symmetric ability, geometric symmetric ability, and mathematics attitude. Cultural symmetry questions (e.g., Fig. 1) were scored out of three: one point for the correct number of rotations, one point for the correct number of reflectional axes, and one point for the correct placement of axes on the image. In cases where students responded with a valid answer (e.g., infinite rotations) but which were incorrect (because the question asked about only one 360 degree rotation), points were awarded when there was consistency of response across questions. A total of fifteen points were possible in this section. Card rotations questions (Fig. 2) and Paper folding questions (Fig. 3) were scored following the guidelines provided by the distributors (Ekstrom et al., 1976). Likert scale attitudinal data (Fig. 4) were scored using the reverse coding method (Field, 2009).

Future data analysis will: assure sampling validity by comparing the performance of the survey population to normative performance on the geometric tasks, search for trends within the card rotations and paper folding test having to do with angular difference (Cooper, 1975) and fold complexity, and establish inter-rater reliability.

Preliminary Results

In the initial sample population we see high variation in symmetric ability in three of the four measures (Fig. 8), indicating that this instrument can parse. Further, there seems to be a positive correlation between symmetric ability and mathematics attitude (Fig. 9).



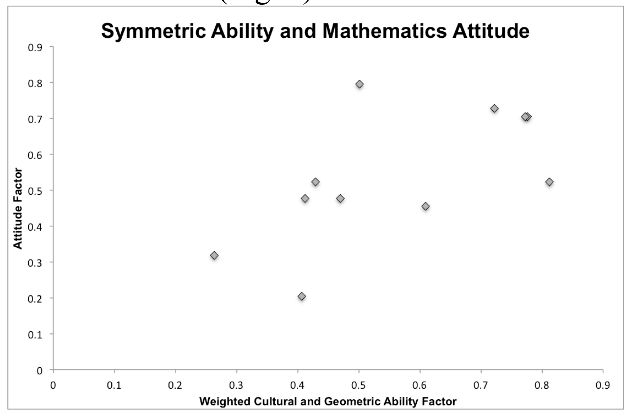


Figure 8. Summary statistics for each section of the symmetric ability survey. Note: scores on the geometric and attitude tasks have been scaled to 15 for comparison purposes.

Figure 9. The data appear to show that as symmetric ability increases (an equal weight given to cultural, card rotations, and paper folding items), so does mathematics attitude depicted as percent ideal response.

Anecdotal and preliminary interview analysis suggests that the cultural items are being interpreted as intended.

Implications and Future Inquiry

Based on preliminary findings it seems that there is a broad range of symmetric abilities among this population of undergraduate students. Further, that having higher levels of symmetric ability may correlate with more positive mathematics attitude. These results suggest that differences may exist between students with high or low symmetric ability. Future research plans include: expanding rigor and sample size of the symmetric ability instrument, and investigating the intersection of symmetric ability with problem solving through interviews. Possible interview findings include: high symmetric ability students prefer but do not natively use symmetric heuristics, low symmetric ability students do not prefer and do not natively use symmetric heuristics, or any combination therein. Further, this line of inquiry will provide a characterization of how students think about symmetry as a heuristic.

Discussion Questions

1. What are your thoughts on the interview tasks? Can you think of other useful tasks to consider?
2. Have you encountered students with high symmetric ability in your own teaching? Did these students have an advantage in your mind?

References

Abas, J. S. (2004). Islamic Geometrical Patterns for the Teaching of Mathematics of Symmetry. *Symmetry in Ethnomathematics*, *12*(1-2), 53–65.

Betz, N. E., & Hackett, G. (1983). The relationship of mathematics self-efficacy expectations to the selection of science-based college majors. *Journal of Vocational Behavior*, *23*(3), 329–345. http://doi.org/10.1016/0001-8791(83)90046-5

Cooper, L. a. (1975). Mental rotation of random two-dimensional shapes. *Cognitive Psychology*, *7*, 20–43. http://doi.org/10.1016/0010-0285(75)90003-1

D'Ambrosio, U. (2001). What is Ethnomathematics, and How Can It Help Children in Schools? *Teaching Children Mathematics*, *7*(6), 308–310.

Drefus, T., Eisenberg, T. (1990). Symmetry in Mathematics Learning. *ZDM.*, *22*(2), 53–59.

Eglash, R., Bennett, A., O'Donnell, C., Jennings, S., & Cintorino, M. (2006). Culturally Situated Design Tools: Ethnocomputing from Field Site to Classroom. *American Anthropologist*, *108*(2), 347–362. http://doi.org/10.1525/aa.2006.108.2.347

Ekstrom, R. B., French, J. W., Harman, H. H., & Derman, D. (1976). *Manual for Kit of Factor-Referenced Cognitive Tests*. Princeton, New Jersey.

Fennema, E., & Sherman, J. (1977). Sex-Related Differences in Mathematics Achievement, Spatial Visualization and Affective Factors. *American Educational Research Journal*, *14*(1), 51–71. http://doi.org/10.3102/00028312014001051

Fennema, E., & Sherman, J. a. (1976). Fennema-Sherman Mathematics Anxiety Scales: Instruments designed to measure attitudes towards the learning of mathematics by females and males. *Journal for Rearch in Mathematics Education*, *7*(5), 324–326. http://doi.org/10.2307/748467

Field, A. P. (2009). *Discovering statistics using SPSS : (and sex and drugs and rock "n" roll)*. London: SAGE.

French, J. W., Ekstrom, R. B., & Price, L. A. (1963). *Manual for Kit of Reference Tests for Cognitive Factors*. Princeton, New Jersey: Educational Testing Service.

Goldin, G. A., & McClintock, C. E. (1980). The Theme of Symmetry in Problem Solving. In S. Krulik & R. E. Reys (Eds.), *Problem Solving in School Mathematics* (pp. 178–194). Reston, Va.: National Council of Teachers of Mathematics.

Holdren, J. P., & Lander, E. (2012). *REPORT TO THE PRESIDENT Engage to Excel: Producing One Million Additional College Graduates with Degrees in Science, Technology, Engineering, and Mathematics*. Retrieved from http://www.whitehouse.gov/sites/default/files/microsites/ostp/pcast-engage-to-excel-v11.pdf

Initiative., C. C. S. S. (2011). Common Core State Standards for Mathematics. Retrieved from http://www.corestandards.org/assets/CCSSI_Math Standards.pdf

Initiative., C. C. S. S. (2012). *Common core state standards for mathematics.* [S.l.]: Common Core State Standards Initiative.

Leikin, R. (2003). Problem-solving preferences of mathematics teachers: Focusing on symmetry. *Journal of Mathematics Teacher Education*, *6*, 297–329.

Leikin, R., Berman, A., & Zaslavsky, O. (2000). Applications of symmetry to problem solving. *International Journal of Mathematical Education in Science and Technology*, *31*(December 2014), 799–809. http://doi.org/10.1080/00207390050203315

Mathematics., N. C. of T. of. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Mulhern, F., & Rae, G. (1998). Development of a Shortened Form of the Fennema-Sherman Mathematics Attitudes Scales. *Educational and Psychological Measurement*, *58*(2), 295–306. http://doi.org/0803973233

Olkun, S. (2003). Making Connections : Improving Spatial Abilities with Engineering Drawing Activities. *International Journal of Mathematics Teaching and Learning*, (April), 1–10. Retrieved from http://www.cimt.plymouth.ac.uk/journal/default.htm

Rizzo, S. (2013). *College Students' Understanding of Geometric Transformations*. University of Maine.

Schoenfeld, A. H. (1987). What's All the Fuss About Metacognition? In *Cognitive Science and Mathematics Education* (pp. 189–215). Hillsdale, NJ: Lawrence Erlbaum Associates.