A Study of Common Student Practices for Determining the Domain and Range of Graphs

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This study focuses on how students in different postsecondary mathematics courses perform on domain and range tasks regarding graphs of functions. Students often focus on notable aspects of a graph and fail to see the graph in its entirety. Many students struggle with piecewise functions, especially those involving horizontal segments. Findings indicate that Calculus I students performed better on domain tasks than students in lower math course students; however, they did not outperform students in lower math courses on range tasks. In general, student performance did not provide evidence of a deep understanding of domain and range.

Keywords: Graphs of functions, Domain and range, Cognitive research

Functions are important because they model quantitative relationships and serve as foundational notions for more advanced mathematics topics (Blair, 2006). However, the concept of a function, the different representations of functions, and how they are linked post challenges for students (Kaput, 1989; Kleiner, 2012; Sierpinska, 1992; Tall & DeMarois, 1996). Domain and range play key roles in understanding relationships between the two variables in a function (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Yet, there has been little research on common practices students use to determine the domain and range, specifically for the graph of a function.

Common Practices: Strategies, Transitional Conceptions, Use of Representations

During the meaning-making process, individuals often rely on their own practices. These practices, which are based on conceptions that have developed around mathematical ideas, include strategies that individuals choose to employ to help develop understanding and solve items. Chiu, Kessel, Moschkovich, and Muñoz-Nuñez (2001) defined a *strategy* as "a sequence of actions used to achieve a goal, such as accomplishing a particular task or solving a particular problem" (p. 219). Following Smith, diSessa, and Roschelle (1993), Moschkovich (1999) defined a *transitional conception* as "a conception that is the result of sense-making, sometimes productive, and has the potential to be refined" (p. 172). To study individuals' meaning-making processes, it is crucial to consider their transitional conceptions along with the strategies they employ and the representations they use when engaged in mathematical tasks.

In previous research (Cho & Moore-Russo, 2014), ten common practices on tasks involving the domain and range of a function's graph were identified. Building on the findings from that study, this study considers the following research questions:

1. How do common student practices align with students' performance on tasks involving the domain and range of a function's graphical representation?
2. How do students in different mathematics courses perform on tasks involving the domain and range of functions in graphical form?

Methods

The study participants were students enrolled in one of three mathematics courses at a four-year college in the eastern United States. Algebraic Problem Solving (APS), Pre-calculus (Precalc), and Calculus I (Calc) courses were selected for this study, since these courses address both the concept of function, in general, as well as the notions of domain and range, in particular. Six of the courses were APS classes, two were Precalc classes, and three were Calc classes. In total, there were 219 participants in the study: 128 APS students (under four instructors), 54 Precalc students (under two instructors), and 37 Calc students (under two instructors).

The APS course, commonly known as College Algebra at other institutions, is open to all students, meets the basic mathematics competency requirement for the college, and introduces the ideas of function, domain, and range. In Precalc, instructors concentrate on how to identify the domain and range of the graphical representations of functions, and students work with a variety of functions, including piecewise functions. In the Calc course, students use the concept of domain and range on graphs, but instructors do not directly teach those concepts.

Data collection

The research team members had over 40 years combined experience teaching secondary and postsecondary courses. Based on their experience and previous research, the researchers developed a paper-and-pencil multiple-choice test that consisted of 20 graphs. The graphs consisted of a variety of functions and included both continuous and discontinuous piecewise functions. Odd numbered items required a response to a function's domain and even numbered items required a response to its range; hence, there were a total of 40 items. Each item had five options. The instrument reliability was acceptable (Cronbach's α = .69).

To remind students of the concepts of functions, domain, and range, the definitions for all three were listed on the front page of the test. Students were motivated to complete this test as a means to check and develop their concepts of domain and range. They did not receive any compensation, and their participation was voluntary. After obtaining consent from student volunteers, the multiple-choice test was administered in class at a time most convenient for the instructors. All students completed the test within 20-30 minutes.

Data analysis

The data were analyzed using SPSS software. All statistical tests used α = .05 when assessing statistical significant and were two-tailed (where appropriate). A MANOVA was used to analyze if students' performance on the domain and range tasks in the 40 items varied according to the college mathematics course in which they were enrolled. This was appropriate since domain and range task performance are both dependent variables in this study. In addition, choosing a MANOVA (as opposed to two separate ANOVAs) reduces the likelihood of committing a Type I error, as well as accounts for any correlation between the dependent variables. In addition, a series of Bonferroni-corrected post hoc comparisons were used to find which math courses differed in domain and range performance.

Results

The first research question examines how students' practices align with their performance on tasks involving the domain and range of functions in their graphical form.

Significant transitional conceptions or strategies

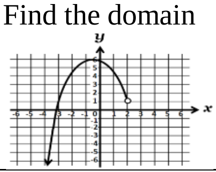
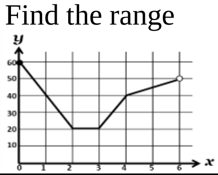
This study looks for relationships between how often students used common conceptions, strategies, or representations in light of their performance on the domain and range items. The research team used the most common student practices identified in previous research (Cho & Moore-Russo, 2014), which are listed in Table 1 with their codes.

Table 1
Common Practices Associated with Incorrect Responses

Abbreviation	Common Practice
EdptFocus	Focusing on the endpoints of a graph or the interval endpoints of a discontinuous graph
ConfuseDR	Confusing the domain and range
IntDescend	Representing an interval in set notation in descending order
NoOverlap	Not combining abutting or overlapping intervals
Intercept	Focusing on either *x*- intercept or *y*-intercepts
IntNotation	Confusing the notations for open ( ) and closed [ ] intervals
RangeLtoH	Treating the range as continuous from the lowest to the highest value for a discontinuous, piecewise function
OpenPoint	Not noticing or ignoring an open point

To show how items were coded for common practices, Table 2 provides examples of the coding used for two items. Note that for each of the 40 items, codes were not assigned to the item's correct answer nor were they assigned to option E, "None of the above."

Table 2
Examples of Codes Assigned to the Options of Selected Items

Test items	Multiple choice options	Assigned Codes
11. Find the domain	A) $(-\infty, 2)$ or $x < 2$	None (correct answer)
	B) $(2, -\infty)$	*EdptFocus, IntDescend*
	C) $\{-3.2\}$ or $x = -3.2$	*Intercept*
	D) $(-\infty, 6)$ or $x < 6$	*ConfuseDR*
	E) None of the above	None
24. Find the range	A) $[60, 20] \cup [20, 20] \cup [20, 40] \cup [40, 50)$	*NoOverlap, IntDescend*
	B) $[20, 60]$ or $20 \leq y \leq 60$	None (correct answer)
	C) $[20, 50) \cup (50, 60]$	*NoOverlap*
	D) $[60, 50)$	*EdptFocus, IntDescend*
	E) None of the above	None

The occurrences of the coded practices for each student were tallied and compared against the percentage of domain and range items the student had answered correctly. The correlation matrix in Table 3 displays those results. Results in Table 3 suggest that the practices students used did not have the same relationship with student performance on the domain and range items on the test. For example, more frequent use of the *Intercept*, *IntNotation*, and *OpenPoint* strategies tended to have more of a negative influence on performance on domain items than range items. Interestingly, neither the *Intercept* nor the *OpenPoint* strategy appeared to be related to students' performance on range items. There were also strategies that had a stronger, negative relationship with performance on range items. The more often students used the *EdptFocus*, *IntDescend*, or *NoOverlap* strategies, the fewer range items they answered correctly. It should be first noted that the *NoOverlap* strategy only applied to range tasks. Next, there does appear to be an issue of multi-collinearity between the *EdptFocus* and the *IntDescend* strategies (r = .94). This is most likely because these strategies were often present in many of the same items. In addition, the lack of a relationship between the *RangeLtoH* strategy with both performance on domain items (r = -.09) and range items (r = -.06) seem to suggest that the use of this strategy neither helps nor hinders a student's performance on domain and range tasks.

Table 3
Summary of Correlations between the Frequencies of Common Practices Associated with Incorrect Responses and Performance on Domain and Range Items

Variable	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
1. % Domain Correct	1									
2. % Range Correct	.63**	1								
3. EdptFocus	-.49**	-.66**	1							
4. ConfuseDR	-.52**	-.50**	.31**	1						
5. IntDescend	-.45**	-.66**	.94**	.34**	1					
6. NoOverlap	-.26**	-.46**	.50**	.11	.47**	1				
7. Intercept	-.28**	.00	.01	.05	.02	-.02	1			
8. IntNotation	-.54**	-.35**	.39**	.19**	.30**	.25**	-.07	1		
9. RangeLtoH	-.09	-.06	-.08	.15*	-.12	-.07	.02	.09	1	
10. OpenPoint	-.25**	.00	-.08	.04	-.19**	-.23**	-.12	.21**	.27**	1

Note. n = 219; *p < .05; **p < .01

Most difficult items for students

The six items with the lowest percentage of correct responses are displayed in Table 4. Items 27, 28, and 29 involved piecewise function graphs with several horizontal segments or open end points. Most students who selected incorrect options for these items either did not notice or ignored the open point to measure the domain or range. Many students also chose option E as the answer for these items, which might suggest they did not have a strategies for how to solve these tasks. Item 34 was a piecewise function graph whose output values overlapped. Most students who selected an incorrect option for this item either did not notice or ignored the overlapped portion. Item 38 was a piecewise function graph with one horizontal segment and two end arrows denoting that the function continued both as the inputs approached negative and positive infinity. Many students seemed to focus on this and selected option C, which stated "all real numbers". However, many students' transitional conceptions failed to take into account the vertical gap in outputs between the values of 2 and 3. The graph displayed in item 17 showed part of a parabola with two open points, one on the x-axis and one on the y-axis. Many students did not notice or ignored the open point on the y-axis.

Table 4
Occurences of Common Practices Associated with Incorrect Responses on Most Difficult Items

Item	Correct Responses	Common Practices Associated with Incorrect Responses						
		EdptFocus	*ConfuseDR*	*IntDescend*	*NoOverlap*	*IntNotation*	*RangeLtoH*	*OpenPoint*
27	20.55%		X		X		X	X
29	22.37%	X					X	X
34	24.20%	X			X		X	
38	24.20%	X	X	X		X		
17	26.03%	X	X	X				X
28	27.85%		X				X	X

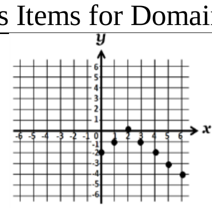
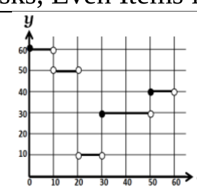
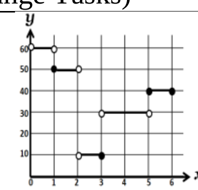
A matched pairs *t*-test was conducted to see if there was any statistically significant difference between how well students performed on the domain items as opposed to the range items. On average, students answered 52.26% of the domain items correctly (SD = 22.16%), whereas they only answered 43.54% of range items correctly (SD = 21.47%). This mean difference of 8.72% was statistically significant, $t(218) = 6.91$, $p < .01$, $d = .47$. On average, students performed nearly a half standard deviation better on the domain items, which is a moderate difference. This result concurred with the previous study's findings, which found that range items were more difficult than domain items for students (Cho & Moore-Russo, 2014).

Performance on items involving piecewise functions

Students seem to lack strategies, even ones related to transitional conceptions, to make meaning of piecewise functions when determining domain and range. Many participants selected option E indicating that none of the response options provided for an item was correct. However, the correct response was located in options A through D for each item on the test. The research team found that a higher percentage of students selecting option E came from piecewise functions that included horizontal segments or disconnected points in their graphs. In fact, the six items with the highest percentages of students choosing E were domain and range tasks related to three graphs of piecewise functions that included horizontal segments or disconnected points. Those items were item 21, 22, 27, 28, 29, and 30 (see Table 5). Recall that odds items involved domain tasks, and even items involved range tasks.

Table 5

Items with Highest Percentages of Option E Responses

		Graph on Test and Associated Items (Odds Items for Domain Tasks, Even Items for Range Tasks)		
		 Items 21 and 22	 Items 27 and 28	 Items 29 and 30
% of Option E Responses	Domain	20.55	19.63	24.20
	Range	23.74	24.20	23.29

The percentages of option E responses was 20.55% for item 21, 23.74% for item 22, 19.63% for item 27, 24.20% for item 28, 24.20% for item 29, and 23.29% for item 30. This could indicate that participants struggled to make meaning of the items involving piecewise function graphs and horizontal segments or points.

Student levels and performance

The second research question considered differences between students in different levels of courses and their performance on domain and range tasks for functions in their graphical forms. The research team found that the level of math had a significant effect on performance on the multiple-choice test, $\Lambda = .90$, $F\ (4, 430) = 5.74$, $p < .01$. From here, the research team decided to examine how class level related to domain and range performance individually. The math course level had a significant effect on domain performance, $F\ (2, 216) = 9.09$, $p < .01$.

Table 6
Results for Items Related to Domain and Range (Reported as Percentages)

Course	Domain		Range	
	M	SD	M	SD
APS	47.70	21.63	40.16	20.32
Precalc	54.91	21.73	49.07	22.13
Calc	64.19	19.91	47.16	22.66
Overall	52.26	22.16	43.54	21.47

As Table 6 illustrates, on average, students who enrolled in Calc had the best domain performance, while those in the APS course had the worst. A series of Bonferroni-corrected post hoc comparisons were run to find which math courses differed in domain performance, and found that the only statistically significant difference in domain performance (using a familywise $\alpha = .05$) occurred between Calc students and APS students. On average, Calc students performed over three-quarters of a standard deviation better than the APS students ($d = .77$). The math course level also had a significant effect on range performance, $F(2, 216) = 4.02$, $p < .05$. On average, Precalc students performed the best on range items (doing better than Calc students), while those in the APS course performed the worst. With the series of Bonferroni-corrected post hoc comparisons, the researchers found that the only statistically significant difference in range performance (again, using a familywise $\alpha = .05$) occurred between Precalc and APS students. As Table 6 illustrates, Precalc students, on average, performed almost 9% better than the APS students; this effect was moderate ($d = .42$).

Discussion and Conclusions

Learning about the domain and range of functions, including studying them in a function's graphical form, is common in many secondary and early postsecondary mathematics courses. Students in Calc should easily make meaning of tasks that involve these topics. Our results, on the other hand, suggest that this is not the case. Overall, many of the study participants seemed to have difficulty performing domain and range tasks on graphs of functions. As Table 6 indicates, the average participant's performance was only 52.26% for domain tasks and 43.54% for range tasks. These findings also confirm what had been noted in previous research that college students, in general, have more difficulty on range tasks than on domain tasks. While our sample illustrates the notion that students' performance on domain tasks tends to improve as they advance to higher level mathematics courses, we did not have statistical support to generalize these findings across the three courses involved. Even though Calc students, on average, had a better understanding of domain than students in the other two math courses; we only had enough evidence to support that students enrolled in Calc had a better understanding of domain than APS students. There was not enough evidence to support the claim that Calc students, on average, had a better understanding of range than APS students or Precalc students.

The relationship between student practices and performance were not the same for the domain and range tasks. As Table 3 suggests, more frequent use of the *Intercept*, *IntNotation*, and *OpenPoint* strategies tended to have a negative influence on performance for domain tasks, while the *EdptFocus*, *IntDescend*, or *NoOverlap* strategies tended to have a negative influence on performance for range tasks. We can conclude that students do not necessarily utilize the same strategies when solving for both domain and range tasks; rather, they discriminate the type of practice they use depending on the need to determine domain or range.

When the participants engaged in the domain and range tasks, many seemed to have traced the graph from the start to the end (i.e., left to the right or bottom to top). Even though the piecewise sections of the functions often abutted or overlapped in their intervals, students seemed to hyper-focus on the "micro" and not the "macro"– forgetting to look at the graph in its entirety and hence failing to combine abutting or overlapping intervals. They also struggled with piecewise functions in ways that suggest that students often fail to take into account the graph as a whole. Just as the saying that a "person can't see a forest for the trees" goes, students might get so involved in identifying notable aspects of a graph (particularly graphs that lack continuity) or tracing a graph through particular points that they fail to see the graph in its entirety. Participants struggled with range items when horizontal segments were part of a discontinuous function's graph. This finding corresponds with the results of previous study and seems to provide evidence that instructors need to recognize that some transitional conceptions students hold need to be revisited to help students make meaning of domain and range at both the micro and macro levels.

Limitations and Future Research

For each item, participants had five response options, including option E "none of the above." Many students selected this response, especially for items with horizontal segments. If the participants had been prompted to write in their answers when selecting the "E" response, they might have more insight to their strategies and transitional conceptions. In addition, we also note that multiple strategies could have been used when students selected a particular choice for each item. This was most likely the reason for the multi-collinearity witnessed in Table 3 between the *EdptFocus* and the *IntDescend* strategies. Hence, there are study limitations that result from the design of the instrument items. Repeated interviews over time with students or longitudinal studies involving pretests and repeated post-tests would provide more detailed insight on how students' transitional conceptions, strategies, and uses of representations develop or persist.

Another limitation relates to the follow-up contrasts conducted for the second research question. We used Bonferroni-corrected post hoc contrasts rather than assume which classes might differ in terms of performance. While our post-hoc contrasts allow us to examine differences between all three class levels, we had to control for the possibility of making a Type I error. Consequently, we may have been too conservative in our findings. Had we determined planned contrasts a priori instead, we might have found more significant differences between math class levels, as our alpha level would have been much higher for determining significance.

Suggestions exist on how high school teachers can emphasize connections while teaching functions (e.g., Moore-Russo & Golzy, 2005), and studies provide evidence that explicit presentation of multiple representations of mathematical ideas and reference to the connections between them using a multimodal approach are important instructional considerations (McGee & Moore-Russo, 2015; Moore-Russo & Viglietti, 2012; Wilmot et al., 2011). Research also suggests that the way ideas related to functions are taught in at the secondary level may vary from the way they are taught at the postsecondary level (Nagle, Moore-Russo, Viglietti, & Martin, 2013). A study of both high school and college instructors could help point out similarities and differences in methods for teaching domain and range and the connections explicitly made during instruction of these topics between the graphical representation and other representations.

References

Blair, R. (2006). *Beyond crossroads—Implementing mathematics standards in the first two years of college*. Memphis, TN: American Mathematical Association of Two-Year Colleges.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*, 352-278.

Chiu, M. M., Kessel, C., Moschkovich, J., & Muñoz-Nuñez, A. (2001). Learning to graph linear functions: A case study of conceptual change. *Cognition and Instruction*, *19*(2), 215–252.

Cho, P., & Moore-Russo, D. (2014). How students come to understand the domain and range for the graphs of functions. *MathAMATYC Educator*, *5*(3), 32-37.

Kaput, J. (1989). Linking representations in the symbol systems of algebra. In S. Wagner & C. Kieran (Eds.), Research issues in the learning and teaching of algebra (pp. 167 – 194). Reston, VA: NCTM

Kleiner, I. (2012). A brief history of the function concept. In *Excursions in the history of mathematics* (pp. 103-124). New York: Springer.

McGee, D., & Moore-Russo, D. (2015). Impact of explicit presentation of slopes in three dimensions on students' understanding of derivatives in multivariable calculus. *International Journal of Science and Mathematics Education*, *13*(2), 357-384.

Moore-Russo, D., & Golzy, J. (2005). Helping students connect functions and their representations. *Mathematics Teacher*, *99*(3), 156-160.

Moore-Russo, D., & Viglietti, J. (2012). Using the K_5 *Connected Cognition Diagram* to analyze teachers' communication and understanding of regions in three-dimensional space. *Journal of Mathematical Behavior*, *31*(2), 235-251.

Moschkovich, J. (1999). Students' use of the *x*-intercept as an instance of a transitional conception. *Educational Studies in Mathematics*, *37*, 169–197.

Nagle, C., Moore-Russo, D., Viglietti, J., & Martin, K. (2013). Calculus students' and instructors' conceptualizations of slope: A comparison across academic levels. *International Journal of Science and Mathematics Education*, *11*(6), 1491-1515.

Sierpinska, A. (1992). On understanding the notion of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 25-28). *Washington, DC:* MAA.

Smith, J., diSessa, A., & Roschelle, J. (1993). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *Journal of the Learning Sciences*, *3*, 115–163.

Tall, D., & De Marois, P. (1996). Facets and layers of the function concept. In L. Puig, & A. Guiterrez (Eds.), *Proceedings of 20th Conference for the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 297-304). Valencia, Spain: PME.

Wilmot, D., Schoenfeld, A., Wilson, M., Champney, D., & Zahner, W. (2011). Validating a learning progression in mathematical functions for college readiness. *Mathematical Thinking and Learning*, *13*(4), 259–291.