Measuring student conceptual understanding: The case of Euler's method

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This preliminary paper reports on early work for a differential equations concept inventory, which is being developed for an NSF-funded project to support mathematics instructors as they implement inquiry-oriented curricula. The goal is to assess student learning of differential equations. Preliminary results show that the iterative method of developing and field testing items, conducting student interviews, and modification may prove successful to complete a valid concept inventory. The field testing and piloting of questions concerning Euler's method show that students do respond as the research suggests but that Euler's method can be recreated by students and the correct response can be "figured out."

Key words: Differential Equations, Concept Inventory, Assessment, Euler's Method

One of the most challenging components of education research is measurement. How do you measure student understanding? How can you tell if your measurements are accurate? In the so-called "hard" sciences, measurement involves a physical instrument, rulers for lengths, scales for weights, etc. Alternatively, education researchers spend a great deal of time and effort both defining and describing their constructs as well as outlining how they can measure those constructs. Along those lines, we are developing an assessment to enable us to measure student understanding of the big ideas in differential equations (DE), as part of a larger study to assess how students taking an inquiry-oriented differential equations course understand the basic concepts of DE compared with students who have taken a traditional DE course. With this report, we discuss this process with a focus on one particular concept: Euler's method to solve a differential equation. Our research question is: How can a written assessment effectively measure students' conceptual understanding in differential equations, particularly the numerical technique called Euler's method?

Literature Review

Assessment

Researchers and instructors have developed a vast array of different types of assessment to aid in measuring learning. Concept inventories are research tools designed to measure learning with special attention paid to their validity and reliability in gauging how students think about the underlying concepts of a subject. There have been a number of concept inventories created in many different academic subjects. Two of the physics assessments, the Mechanics Diagnostic Test (Halloun & Hestenes, 1985) and the Force Concept Inventory (Hestenes, Wells, & Swackhammer, 1992) have been influential in the development of mathematics concept inventories. Common themes include the use of student interviews in the validation process and the iterative nature of writing and revising both the taxonomy and assessment items. Following from the two physics concept inventories, two foundational mathematical concept inventories are described below, the Precalculus Concept Assessment (Carlson, Oehrtman, & Engelke, 2010) and the Calculus Concept Inventory (Epstein, 2007). In our review of these assessments, we identified four primary steps in developing a concept inventory: (1) deciding what concepts to cover, i.e. the taxonomy of the assessment, (2) writing the assessment items, (3) validating the assessment items, and (4) validating the assessment as a whole.

Calculus Concept Inventory

The Calculus Concept Inventory (CCI) was designed and validated based on the Force Concept Inventory: the major concepts to be assessed were outlined, the items were written by a team with knowledge of the content, then items were reviewed using clinical interviews or what Epstein calls "cognitive laboratories" (2007, p. 167). Finally, a cyclic process of revision and analysis took place. In their first pilot of about 250 students, Epstein noted that scores were near the random guess level, which led to significant modifications of the items, specifically making them much easier. The CCI continues to be in use, but there has not been a great deal of studies published about the results.

Precalculus Concept Assessment Inventory

Carlson et al. (2010) provide great depth on the creation of the Precalculus Concept Assessment (PCA). First, the researchers developed a 34 item, open-ended assessment for the purposes of investigating students understanding of function. The results of the administration of this assessment came to form a Function Framework, which would serve as an initial draft of the PCA Taxonomy. The same process as the other concept inventories continued and the developers are validating the PCA currently.

Selected research on student understanding of differential equations

Analytical Solution Strategies

Researchers have found that students overwhelmingly elect analytical solution strategies when prompted to solve differential equations (Arslan, 2010; Camacho-Machin, Perdomo-Diaz, & Santos-Trigo, 2012a; Habre, 2003; Rasmussen, 2001). However, while students are relatively successful in finding solutions using various analytical solution techniques, they struggle to identify the appropriate strategy for various problem types (Camacho-Machin et al., 2012a) and that ability in solving DEs analytically does not necessarily imply deeper conceptual understanding (Arslan, 2010).

Graphical Solution Strategies and Representations

Students tend to devalue graphical representations of both the DE and their solutions instead relying heavily on analytical techniques and algebraic representations (Habre, 2003; Rasmussen, 2001; Trigueros, 2001). Specifically, researchers have shown that students have trouble both understanding graphical representations as well as constructing them (Camacho-Machin et al., 2012a; Camacho-Machin, Perdomo-Diaz, & Santos-Trigo, 2012b; Rasmussen, 2001). Fortunately, a few studies have provided evidence that students can use graphical representations productively, usually after being prompted (Habre, 2003), and that certain instructional strategies have been shown to help students retain more knowledge concerning graphical representations (Habre, 2003; Kwon, Rasmussen, & Keene, 2005).

Numerical Solution Strategies

Very little educational research exists on student understanding of Euler's method. Rasmussen (2001) characterizes students' conceptions of approximate solutions in three ways: (1) "A numerical approximation inscribes the exact solution," (2) "A numerical approximation 'tracks' the exact solution by using the slope of the exact solution at each step in the approximation," and (3) "A numerical approximation 'tracks' the exact solution via nearby solutions" (p. 76). Furthermore, Rasmussen describes how students' ideas about other approximation methods in mathematics (e.g., Riemann sums and the definite integral) may play a role in how students think about numerical approximations in DEs. These potential mental images of the relationship between approximate and exact solutions informed the selection of multiple-choice items in our concept inventory (see Selected Findings).

Theoretical Framework

Some research and writing to develop a version of a concept inventory for differential equations exists. This earlier work uses the Relational Understanding of Procedures

Framework to address how knowledge in DE may be constructed (Keene, Glass, & Kim, 2011). The primary categories in this framework are: anticipate the outcome, identify the correct procedure, correctly use the procedure, understand the "whys" of the procedure, verify the solution graphically and symbolically, and make connections across the representations involved. An assessment was developed by mathematics educators and piloted in 2008. The assessment was comprised of 30 questions that all related to two analytic solution techniques and one numerical solution technique (Euler's method). Field testing was conducted and revisions were made. Our current work is framed by and directly utilizes this earlier research and framework.

Methods

In our review of the pre-existing concept inventories, we identified four primary steps in developing a concept inventory, (1) deciding what concepts to cover, i.e. the taxonomy of the assessment, (2) writing the assessment items, (3) validating the assessment items, and (4) validating the assessment as a whole. Therefore, in order to create the differential equations concept inventory (DECI), we first had to decide on what topics and concepts should be covered by the assessment and then write or compile the items we felt best assessed students' knowledge. To create the list of topics and concepts, also referred to as the taxonomy for the assessment, we completed a syllabus analysis of various DE courses, spoke with experts, and pulled from our own expertise informed by our experience as well as our work with the existing literature on differential equations assessments and student thinking.

Syllabus analysis

We investigated a sampling of ten DE courses from across the United States, including a wide range of universities. We selected a mix of public and private universities (six and four respectively), various sizes (smallest: approximately 3,200 undergraduates, largest: approximately 46,000), and a range of different primary textbooks (six different textbooks from the ten chosen courses). From these syllabi, we created a list of topics covered by each course and then aligned them to generate a list of any topic or concept that were listed frequently. Even though not complete, this did provide a useful reference in deciding what specific topics to include on the assessment and was a useful store of information when drafting the taxonomy of the assessment.

Taxonomy

We constructed the taxonomy for the DECI by beginning with the relational understanding framework and then referencing results from a syllabi analysis, discussions with experts on DEs, the cognitive research on DEs, the list of topics from the inquiry-oriented differential equations course, and the research teams' experience. We started by using the overarching themes discussed by DE experts as well as the topics covered on the relational understanding framework from Keene, Glass, & Kim (2011). From here, we looked at the major topics with considerable overlap from the syllabus analysis (including the inquiry-oriented differential equations course) to ensure we were hitting a number of the important topics from standard DE courses. This taxonomy underwent significant revision as we gathered evidence of student thinking on the various items through the task-based interviews. Space does not permit us to publish the taxonomy here.

Field testing data collection

In Spring 2015, we piloted a selection of possible items in two DE classrooms. Two differential equations teachers (referred to herein as classes A and B) who were participating in an online workgroup for an NSF-funded project to investigate instructional change agreed to use some of the potential assessment items in their introductory DE class. Class A was an introductory DE course of 15 students using inquiry-oriented materials at a small public liberal arts college in the eastern United States and they responded to the items on the final exam. Class B consisted of 20 students at a private university in the southern United States and responded to the items in an out of class assignment. This course focused on more traditional procedures and proof.

In Summer 2015, we conducted task-based interviews with five students in a summer section of DEs at a large research university in the southeastern United States. During the interviews, students were asked to work on assessment items in front of the researcher and to think aloud about their process. The interviewer asked probing questions while trying to minimize the introduction of any new mathematical concepts or vocabulary. Questions were selected for the interview protocol based on the preliminary findings of the field test described earlier and included both multiple choice and open-ended items. Interviews were audio and video recorded, and all written work was collected and scanned as PDFs for preservation and analysis.

Data analysis

Written Assessment

During the spring administration of the DECI, there were both open-ended and multiple choice formatted items. In this report we are only discussing multiple-choice items and so we will focus our discussion of analysis methods to those items. For the multiple-choice items, analysis consisted primarily of investigating the appropriateness of the answer choices and the difficulty of each item. The data were entered into a spreadsheet where each participant's answer choices were included for each question. Two separate analyses were carried out on these items, the first concerning how many students answered the question correctly and the second concerning how many students chose each of the provided distractors. In terms of difficulty, we looked to determine if any of the questions were either too challenging or too easy.

After the initial analysis for difficulty, analysis followed on the students who answered questions incorrectly. Primarily, the goal of such analysis was to ensure that the distractors were working effectively and that each was being chosen at least some of the time. In previous work, assessment authors have marked any distractors that were not chosen at least 5% of the time for potential revision (Carlson et al., 2010) and so this rule was our initial guide for throwing out distractors.

Task-Based Interviews

In the near future, we will be conducting an analysis of the task-based interviews in order to continue validating the assessment items. Identifying the ways in which students thought about the items will provide us with evidence that the items are actually measuring the concepts we assume they are measuring. On multiple-choice items, our primary goal will be to investigate what knowledge students attend to as they complete the items. To do this, we will employ an open-coding strategy on the video data in an attempt to outline the big ideas from DEs that students are attending to as they work through the problems.

Selected Findings: Euler's Method Question

For this preliminary report, we focus on one particular question that was administered to both pilot groups and used in the task-based interview (Figure 1).

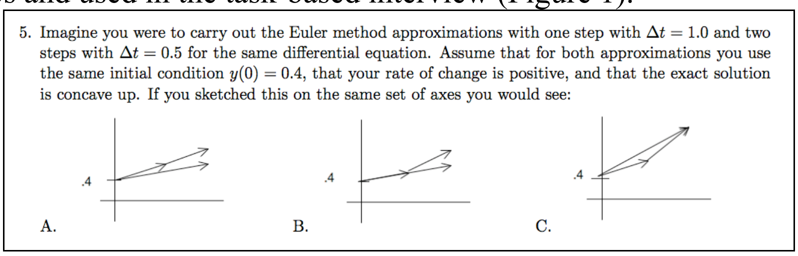

Figure 1. Euler's question.

The correct answer to this question is B. In using Euler's method, one uses the value of the derivative (rate of change) at one point to create a line segment for a defined constant change in the independent variable (in this case *t*). You then reevaluate the derivative (rate of change) and create a new line segment.

The results from the two classes and the interviews were particularly interesting and informative (see Figure 2). For the inquiry oriented class (*n=14*), the majority of the students answered correctly. We posit this is because this may have been explicitly discussed in the course, as the materials focus on the understanding of Euler's method. The traditional class (*n=14*) answered primarily C. This aligns with Rasmussen's (2001) findings on approximations, specifically that when doing the approximations, students want the lines to "track" the actual solution.

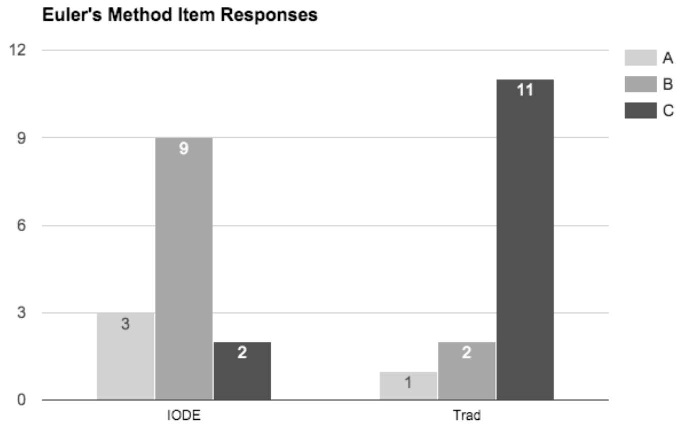

Figure 2. Results of choices for the two classes.

When the researcher asked this question of the five interview students, we found out more interesting information. The students in the interview had not ever seen Euler's method before so initially they did not know how to answer. However, two of the students were able to use information presented in the problem context with their knowledge of other approximation methods (e.g., Taylor series approximations of functions) to reason their way through to the correct answer, just as Rasmussen (2001) discussed. Afterwards, they were still not confident they were correct but had, in that moment, recreated Euler's method for approximating solutions to DEs.

Conclusion

The analysis and field testing of this assessment will continue during the next two years, but we have found that this method of developing a concept inventory seems to be useful. We know that the distractors on this question and others need to be revisited to make sure they are effective. We intend to continue with this same work, even though it is very time consuming. Thus far, we have found interest in the DECI to be high and we would like to include the RUME community in the continuation of this work. To this end, we will ask the following questions in the presentation:

1. What alternate conceptions do you see when teaching your students techniques to solve differential equations?
2. What do you consider the most important conceptions students need to develop in differential equations?
3. Do you know of alternatives to concept inventories to help assess student learning?

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