Investigating the role of a secondary teacher's image of instructional constraints on his enacted subject matter knowledge

Michael A. Tallman
Oklahoma State University

I present the results of a study designed to determine if there were incongruities between a secondary teacher's mathematical knowledge and the mathematical knowledge he leveraged in the context of teaching, and if so, to ascertain how the teacher's enacted subject matter knowledge was conditioned by his conscious responses to the circumstances he appraised as constraints on his practice. To address this focus, I conducted three semi-structured clinical interviews that elicited the teacher's rationale for instructional occasions in which the mathematical ways of understanding he conveyed in his teaching differed from the ways of understanding he demonstrated during a series of task-based clinical interviews. My analysis revealed that that the occasions in which the teacher conveyed/demonstrated inconsistent ways of understanding were not occasioned by his reacting to instructional constraints, but were instead a consequence of his unawareness of the mental activity involved in constructing particular ways of understanding mathematical ideas.

Key words: Mathematical Knowledge for Teaching; Enacted Knowledge; Instructional Constraints; Secondary Mathematics.

Introduction

Mathematics educators have devoted increased attention in recent years to: (1) identifying categories of knowledge that teachers must possess to effectively support students' mathematics learning (e.g., Ball, 1990; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Hill & Ball, 2004; Hill, Ball, & Schilling, 2008; Hill, Schilling, & Ball, 2004; Shulman, 1986, 1887), (2) characterizing the specific mathematical ways of understanding that allow teachers to engage students in meaningful learning experiences (e.g., Ma, 1999; Yoon et al., 2014), (3) discerning how teachers might construct such ways of understanding (e.g., Harel, 2008; Harel & Lim, 2004; Silverman & Thompson, 2008; Tallman, 2015), and (4) developing instruments to measure teacher's knowledge and its effect (e.g., Hill et al., 2008; Hill, Rowan, & Ball, 2005; Thompson, 2015). These initiatives, while essential to the enterprise of improving students' mathematics learning, do not ensure teachers will utilize the full extent of their knowledge in the act of teaching. Teachers must recognize the knowledge they possess as appropriate to employ in the process of achieving their goals and objectives in the context of practice. This recognition is subject to a host of cognitive and affective processes that have thus far not been a central focus of research on teacher knowledge in mathematics education (Day & Qing, 2009, p. 17; Hargreaves & Shirley, 2009, p. 94; Meyer, 2009, p. 89; Schutz et al., 2009, p. 207). Identifying the factors that condition the knowledge teachers utilize in the context of teaching, and understanding the effect of such factors on the quality of teachers' enacted knowledge, is imperative for improving students' mathematics learning and for fashioning well-informed teacher preparation and professional development programs and educational policies that take seriously the effect of teacher knowledge and the factors that compromise it. It is to this end that the present paper seeks to contribute. In particular, I address the following research questions:

RQ1: Are there incongruities between a teacher's subject matter knowledge and his enacted subject matter knowledge? [1]

RQ2a: If so, in what ways does the teacher's image of instructional constraints condition the subject matter knowledge he utilizes while teaching?

RQ2b: If not, how is the teacher appraising and/or managing what he perceives as instructional constraints so that these constraints do not condition the mathematical knowledge he enacts while teaching?

Theoretical Framing

The "image of" qualifier in the title of this paper suggests my radical constructivist approach to defining instructional constraints. I take the position that environmental circumstances per se in the absence of a teacher's construal of them cannot constrain his or her practice, but the teacher's construction and appraisal of environmental circumstances can and often does. For this reason, I contend that particular circumstances cannot maintain an ontological designation as instructional constraints, however consensual are teachers' construction and appraisal of such circumstances. Therefore, in consonance with radical constructivism's skeptical position on reality, I define instructional constraints as an individual teacher's subjective construction of the circumstances that impede the teacher's capacity to achieve his or her instructional goals and objectives. Such subjective constructions are the only "constraints" that maintain the potential to influence teachers' instructional actions. Accordingly, I locate instructional constraints in the mind of individuals, not the environment. This conceptualization stands in stark contrast to the common perception of instructional constraints as external pressures that exert influence on the quality of teachers' instruction. According to this view, the pressure comes from without instead of from within. My interest in understanding how a secondary teacher's image of instructional constraints conditioned the mathematical ways of understanding and ways of thinking he utilized in the context of teaching necessitated my constructing a model of the teacher's construction of those circumstances he appraised as constraints on his practice.

As a result of my view that instructional constraints are subjective constructions that reside in the minds of teachers, I consider anything that a teacher appraises as an imposition to achieving his or her instructional goals and objectives to be an instructional constraint. The appraisal need not even be of an external circumstance. A teacher may appraise internal characteristics such as his or her mathematical self-efficacy, social endowments, creativity, tolerance, attitude, perseverance, temperament, empathy, confidence, etc., as imposing limits on the quality of his or her instruction. Since a teacher's appraisal of such intrinsic characteristics is a subjective construction in the same way that a teacher's appraisal of external circumstances is, both types of appraisals have the capacity to influence teachers' practice in the same way.

[1] I note that the identification of incongruities between the teacher's subject matter knowledge and the subject matter knowledge he invokes while teaching is from my perspective. Similarly, characterizing the effect of a teacher's image of instructional constraints on his enacted mathematical knowledge is also a characterization from my perspective.

Methods

My experimental methods proceeded in three phases. In the first phase, I conducted a series of nine task-based clinical interviews (Clement, 2000; Goldin, 1997; Hunting, 1997) that allowed me to construct a model of the participating teacher's (David's) mathematical knowledge of various topics associated with trigonometric functions. In the second data collection phase, I used video data from 37 classroom observations to construct a model of the mathematical knowledge David utilized in the context of classroom practice. Finally, I employed a phase of three semi-structured clinical interviews to construct a model of David's perception of instructional constraints and to discern the role of this image on the quality of his enacted mathematical knowledge.

The goal of the series of task-based clinical interviews was to facilitate my construction of a model of David's *ways of understanding* and *ways of thinking* (Harel, 2008) relative to angle measure, the outputs and graphical representation of sine and cosine, and the period of sine and cosine. Constructing a model of an individual's cognition by projecting or imputing one's cognitive schemes to the individual constitutes developing a *first-order model* (Steffe & Thompson, 2000). This is in contrast to developing a *second-order model*, in which the researcher attempts to make sense of the individual's actions by interpreting them through the lens of his or her model of the individual, not through his or her own cognitive schemes (ibid.). It is important to note that the goal of the series of task-based clinical interviews I conducted was to construct a second-order model of David's mathematical knowledge. Although I constructed a second-order model of David's mathematics, this model does not constitute a direct representation of David's knowledge, but rather a viable characterization of plausible mental activity from which his language and observable actions may have derived. Constructing such a model involved my generating prior to, within, and among task-based clinical interviews tentative hypotheses of David's ways of understanding that explained my interpretation of the observable products of his reasoning. I developed these provisional hypotheses by attending to David's language and actions and abductively postulating the meanings that may lie behind them. I designed and modified tasks for subsequent interviews to test, extend, articulate, and refine my tentative hypotheses of David's mathematical knowledge.

All task-based clinical interviews took place in David's classroom after school on the days that best suited his schedule. I attempted to schedule the interviews so that there was at least one day between each to accommodate for ongoing analysis, and accomplished this with the exception of the last two task-based clinical interviews. In each interview, I obtained video recordings that captured David's writing, expressions, and gestures. I also created videos of my computer screen via *QuickTime Player* to capture the *didactic objects* (Thompson, 2002) David and I discussed as well as any work David completed on the computer. Additionally, I collected and scanned all written work that David produced during the interviews.

I collected daily video recordings of two of David's Honors Algebra II class sessions over a seven-and-a-half-week period, which resulted in 37 videos of classroom teaching. The only days I did not intend to collect videos of David's teaching were those days students were testing or the days David was teaching content unrelated to the angle measure, sine, or cosine. While the classroom observations did not demand the type of ongoing analysis that was part and parcel of the series of task-based clinical interviews, I documented, in the form of memos, the mathematical understandings and ways of thinking David afforded his students the opportunity to construct. I must emphasize that I characterized the ways of understanding and ways of thinking David *allowed* his students to construct, and not the understandings and ways of thinking his students *actually* constructed. In essence, I documented the understandings that I would be able to construct, and the ways of thinking that I would be able to develop, were I an engaged student in the class with sufficient background knowledge, uninhibited by unproductive understandings or disadvantageous ways of thinking.

The objective of the third phase of my experimental methodology was to obtain data that allowed me to construct a model of David's image of those aspects of his environmental context that he appraised as constraints on the quality of his instruction, and to determine the way in which this image conditioned the mathematical knowledge he employed in the context of teaching. Constructing such a model and determining the effect that David's image of instructional constraints had on his enacted subject matter knowledge involved my conducting a series of three semi-structured clinical interviews after David completed his instruction of trigonometric functions.

The content of these semi-structured clinical interviews was heavily informed by my analysis of the data I obtained from the series of task-based clinical interviews as well as from David's teaching. Based on my analysis of this data, I selected video clips to discuss with David during the clinical interview sessions to discern the role of David's image of instructional constraints on the quality of his enacted mathematical knowledge. I devoted particular attention to ascertaining David's rationale for those instructional actions in which the mathematics he allowed students to construct differed from the mathematical ways of understanding he demonstrated during the series of task-based clinical interviews. It is essential to point out that I did not assume David recognized the discrepancies I noticed in the videos excerpts I selected to discuss. Therefore, after having presented pairs of videos to David that I believed demonstrated him conveying/supporting discrepant meanings, I asked him to compare the ways of understanding he communicated in both videos. My rationale for doing so was to determine if David recognized the same inconsistencies that I noticed in the ways of understanding he demonstrated/conveyed.

Analytical Framework

I leveraged explicit formalizations of *quantitative reasoning* (Smith & Thompson, 2007; Thompson, 1990, 2011) in the design of the present study and my analysis of its data. A growing body of research (e.g., Castillo-Garsow, 2010; Ellis, 2007; Moore, 2012, 2014; Moore & Carlson, 2012; Oehrtman, Carlson, & Thompson, 2008; Thompson 1994, 2011) has identified quantitative reasoning as a particularly advantageous way of thinking for supporting students' learning of a wide variety of pre- and post-secondary mathematics concepts. Additionally, this body of research has demonstrated the diagnostic and explanatory utility of quantitative reasoning as a theory for how one may conceptualize quantitative situations.

Quantitative reasoning is a characterization of the mental actions involved in conceptualizing situations in terms of *quantities* and *quantitative relationships*. A *quantity* is an attribute, or quality, of an object that admits a measurement process (Thompson, 1990). One has conceptualized a quantity when she has identified a particular quality of an object and has in mind a process by which she may assign a numerical value to this quality in an appropriate unit (Thompson, 1994). It is important to note that quantities do not reside in objects or situations, but are instead constructed in the mind of an individual perceiving and interpreting an object or situation. Quantities are therefore conceptual entities (Thompson, 2011).

Conceptualizing a quantity does not require that one assign a numerical value to a particular attribute of an object. Instead, it is sufficient to simply have a measurement process in mind and to have conceived, either implicitly or explicitly, an appropriate unit. *Quantification* is the process by which one assigns numerical values to some quality of an object (Thompson, 1990). Note that one need not engage in a quantification process in order to have conceived a quantity, but must have in mind a quantification process whereby she may assign numerical values to the quantity (Thompson, 1994). Defining a process by which one may assign numerical values to a quantity often involves an operation on two other quantities. In such cases we say that the new quantity results from a *quantitative operation*—its conception involved an operation on two other quantities. Quantitative operations result in a conception of a single quantity while also defining the relationship among the quantity produced and the quantities operated upon to produce it (Thompson, 1990, p. 12). It is for this reason that quantitative operations assist in one's comprehension of a situation (Thompson, 1994). It is important to note the distinction between a quantitative operation and a numerical or *arithmetic operation*. Arithmetic operations are used to calculate a quantity's value whereas quantitative operations define the relationship between a new quantity and the quantities operated upon to conceive it (Thompson, 1990).

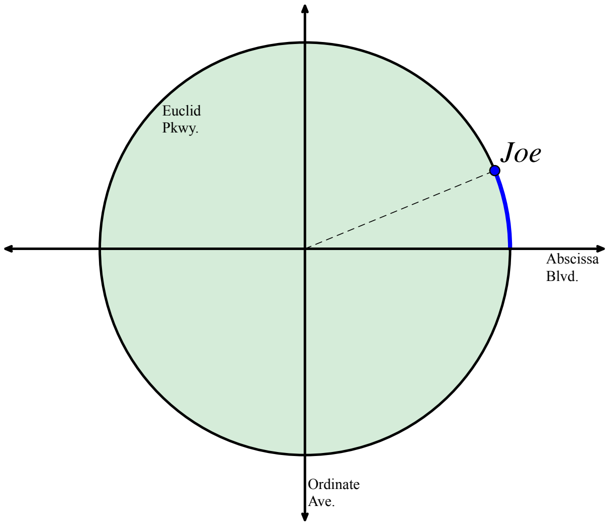
Results

On several occasions David demonstrated ways of understanding during the series of task-based clinical interviews that were inconsistent or incompatible with the ways of understanding his instruction supported. I selected three such occasions to discuss with David during a phase of clinical interviews I conducted after David completed his instruction of trigonometric functions. Specifically, I presented David with three pairs of videos, each containing an excerpt from the series of task-based clinical interviews and an excerpt from his classroom teaching. From my perspective, these pairs of videos exemplified David communicating discrepant mathematical meanings. My purpose in presenting David with these pairs of videos was to determine if he willingly compromised the quality of his enacted mathematical knowledge in response to the circumstances and events he appraised as instructional constraints. The following is a presentation of my analysis of our conversation around one of these three pairs of video excerpts. I do not discuss my analysis of David's and my conversation around the second and third pair of video excerpts since the conclusions drawn therefrom are consistent with those I present below.

I presented David with a video from the fourth task-based clinical interview in which he used an applet (see Table 1) to successfully approximate the values of sin(0.5) and cos(¾). During this interview David interpreted the task of approximating the value of sin(0.5) as, "Estimate how many radius lengths is Joe north of Abscissa Boulevard when the angle traced out by his path is 0.5 radians." In particular, David interpreted the 0.5 as representing the number of radius lengths that Joe had traveled along Euclid Parkway and sin(0.5) as representing Joe's distance north of Abscissa Boulevard in units of radius lengths. David similarly interpreted the task of approximating the value of cos(¾) in the following way: "Estimate how many radius lengths Joe is to the east of Ordinate Avenue when his path has traversed an arc that is $3/4^{\text{ths}}$ times as long as the radius of Flatville." David's response to the task of using the applet in Table 1 to approximate the values of sin(0.5) and cos(¾) suggests that he had constructed the outputs of sine and cosine as quantities; that is, as measurable attributes of a geometric object. After David watched the video except from the fourth task-based clinical interview, he described the way of understanding he demonstrated in a way that was consistent with my interpretation.

Table 1
Applet Designed to Support a Quantitative Understanding of Sine and Cosine Values

Suppose Joe is riding his bike on Euclid Parkway, a perfectly circular road that defines the city limits of Flatville. Ordinate Avenue is a road running vertically (north and south) through the center of Flatville and Abscissa Boulevard is a road running horizontally (east and west) through the center of Flatville. Assume Joe begins riding his bike at the east intersection of Euclid Parkway and Abscissa Boulevard in the counterclockwise direction.



After David watched the video excerpt from the fourth task-based clinical interview, I presented him with a video excerpt from Lesson 7 (which occurred four days after the fourth task-based clinical interview) in which he defined the outputs of sine and cosine relative to the following two cases: (1) when the radius of the circle centered at the vertex of an angle has a measure of one unit and (2) when this radius does not have a measure of one unit. Specifically, in the video excerpt David claimed that if the radius of the circle has a measure of one unit, then the sine and cosine values of the angle's measure are respectively equal to the y- and x-coordinates of the terminus of the subtended arc. David then explained that if the radius of the circle centered at the angle's vertex does not have a measure of one unit, then the values of sine and cosine are given by the respective ratios of the y- and x-coordinate of the terminal point to the length of the radius. It is noteworthy that David's explanation did *not* support students in conceptualizing sine and cosine values as the measure of a quantity in a particular unit. In other words, David's explanation in Lesson 7 did not support students in being able to answer the question, "What are the attributes to which sine and cosine values may respectively be applied as measures and in what unit are these attributes being measured?" In contrast to the quantitative way of understanding the outputs of sine and cosine David demonstrated in the fourth task-based clinical interview, during Lesson 7 David conveyed sine and cosine values as respectively representing y- and x-coordinates of the terminal point, or as arithmetic operations (i.e., $\sin(\theta) = y/r$ and $\cos(\theta) = x/r$).

After David viewed the two video excerpts, I asked him to determine if the way of understanding he supported in the excerpt from Lesson 7 differed from the understanding he employed to approximate the value of sin(0.5) and cos(¾) in the excerpt from the fourth task-based clinical interview (Excerpt 1).

Excerpt 1

Michael:	Is there any way that the understanding of sine and cosine you convey in this clip (*Lesson 7*) is different from what you did here (*Task-Based Clinical Interview 4*)?
David:	Only in the units of measure that we started with to obtain the ratio, but in the end we end up with an output that is a proportion of the entire radius. So in the end, no [they aren't different]. In the end they end up giving me the same thing.

David did not appear to recognize the way of understanding he demonstrated in the first video excerpt as being fundamentally different from the way of understanding he conveyed in the second. David's remark in Excerpt 1 focused primarily on the outcome of his application of two discrepant (from my perspective) ways of understanding instead of attending to the ways of understanding themselves. Like several occasions in other interviews in which David demonstrated an incapacity to attend to ways of understanding—either his own or his students'—his remarks in Excerpt 1 demonstrate that he had not achieved clarity relative to the mental activity involved in his own ways of understanding, nor of those he intended to support in his teaching. Had David done so, he would likely have been positioned to notice the discrepant meanings he conveyed in the videos I presented. David similarly failed to identify the inconsistent meanings he communicated in the other two pairs of video excerpts I presented to him.

Discussion

To investigate the role of David's image of instructional constraints on his enacted subject matter knowledge, I provided opportunities for him to rationalize occasions in which the ways of understanding he supported in his teaching differed from the ways of understanding he demonstrated during a series of task-based clinical interviews. My analysis of our conversation around all three pairs of video excerpts revealed that David failed to notice the discrepancy in the ways of understanding he conveyed/demonstrated in these excerpts. David's inability to recognize such discrepancies suggests that he was not consciously aware of the mental actions that comprise the meanings he intended to promote in his teaching, as such awareness would likely have equipped David with the cognitive schemes to recognize the inconsistent and often incompatible ways of understanding he conveyed in the excerpts we discussed. My analysis further revealed that the occasions in which David conveyed/demonstrated discrepant ways of understanding were not occasioned by his reacting to his image of instructional constraints.

The results of this study suggest that inconsistencies between mathematics teachers' subject matter knowledge and their enacted subject matter knowledge do not necessarily result from teachers' making conscious concessions to the quality of their enacted knowledge in the process of accommodating for the circumstances and events they appraise as constraints on their practice. Such inconsistencies may be a byproduct of teachers' unawareness of the mental activity that constitute their ways of understanding mathematical ideas. Pre-service mathematics teacher educators and in-service professional development specialists should therefore take care to provide opportunities for teachers to have explicit answers to questions like, "When my students read the symbols '$\sin(\theta)$' what do I want them to imagine?" and "When my students look at an angle and think about measuring it in radians, what do I want them to visualize in their minds?" Providing opportunities for teachers to achieve such conscious awareness of the mental activity involved in particular ways of understanding may minimize the potential that teachers will not leverage the full extent of their subject matter knowledge to support students' mathematics learning.

References

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal, 90*, 449-466.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade and how can we decide? *American Educator* 14-22; 43-46.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389-407.

Castillo-Garsow, C. (2010). *Teaching the Verhulst model: A teaching experiment in covariational reasoning and exponential growth.* Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In R. Lesh & A. Kelly (Eds.), *Handbook of Research Methodologies for Science and Mathematics Education* (pp. 547-589). Hillsdale, NJ: Lawerence Erlbaum.

Day, C., & Qing, G. (2009). Teacher emotions: Well being and effectiveness In P. A. Schutz & M. Zembylas (Eds.), *Advances in teacher emotion research* (pp. 15-31): New York: Springer.

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction, 25*(4), 439 - 478.

Goldin, G. A. (1997). Observing Mathematical Problem Solving through Task-Based Interviews. *Journal for Research in Mathematics Education. Monograph, 9*, 40-177.

Harel, G. (2008). What is mathematics? A pedagogical answer to a philosophical question. In R. B. Gold & R. Simons (Eds.), *Current Issues in the Philosophy of Mathematics From the Perspective of Mathematicians*: Mathematical Association of America.

Harel, G., & Lim, K. (2004). *Mathematics teachers' knowledge base: Preliminary results.* Paper presented at the 28th Conference of the International Group for the Psychology of Mathematics Education, Bergen, Norway.

Hargreaves, A., & Shirley, D. (2009). *The fourth way: The inspiring future for educational change*. Sage Publications.

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education, 35*(5), 330-351.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.

Hill, H. C., Blunk, M. L., Charalambos, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*(4), 430-511.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal, 105*(1), 11-30.

Hunting, R. (1997). Clinical interview methods in mathematics education research and practice. *Journal of Mathematical Behavior, 16*(2), 145-165.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Meyer, D. K. (2009). Entering the emotional practices of teaching. In P. A. Schutz & M. Zembylas (Eds.), *Advances in teacher emotion research* (pp. 73-91): New York: Springer.

Moore, K. C. (2012). Coherence, quantitative reasoning, and the trigonometry of students. . In R. Mayes & L. L. Hatfield (Eds.), *Quantitative Reasoning and Mathematical Modeling: A Driver for STEM Integrated Education and Teaching in Context* (pp. 75-92). Laramie, WY: University of Wyoming.

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education, 45*(1), 102-138.

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *Journal of Mathematical Behavior, 31*(1), 48-59.

Oehrtman, M., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M.P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp.150-171). Washington, D.C.: Mathematical Association of America.

Schutz, P. A., Aultman, L. P., & Williams-Johnson, M. R. (2009). Educational psychology perspectives on teachers' emotions In P. A. Schutz & M. Zembylas (Eds.), *Advances in teacher emotion research* (pp. 195-212): New York: Springer.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-22.

Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11*, 499-511.

Smith, J., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95-132). New York: Erlbaum.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillsdale, NJ: Erlbaum.

Tallman, M. (2015). *An examination of the effect of a secondary teacher's image of instructional constraints on his enacted subject matter knowledge*. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University.

Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebra*. Center for Research in Mathematics & Science Education: San Diego State University.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: SUNY Press.

Thompson, P. W. (2002). Didactic objects and didactic models in radical constructivism. In K. Gravvemeijer, R. Lehrer, B. v. Oers, & L. Verschaffel (Eds.), *Symbolizing, modeling, and tool use in mathematics education* (pp. 191-212). Dordrecht, The Netherlands: Kluwer.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sepulveda (Eds.), *Plenary paper presented the annual meeting of the International*

Group for the Psychology of Mathematics Education, (Vol. 1, pp. 31-49). Morelia, Mexico: PME.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education* (pp. 33-57). Laramie, WY: University of Wyoming.

Thompson, P. W. (2015). Researching mathematical meanings for teaching. In L. D. English & D. Kirshner (Eds.), *Third handbook of international research in mathematics education* (pp. 435-461). New York: Taylor & Francis.

Yoon, H., Byerley, C., & Thompson, P. W. (in press). Teachers' meanings for average rate of change in U.S.A. and Korea. In T. Fukawa-Connelly (Ed.), *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education*. Pittsburgh, PA: RUME.