Results from the Group Concept Inventory: Exploring the Role of Binary Operation in Introductory Group Theory Task Performance

Kathleen Melhuish
Teachers Development Group

Jodi Fasteen
Carroll College

Binary operations are an essential, but often overlooked topic in advanced mathematics. We present results related to student understanding of operation from the Group Concept Inventory, a conceptually focused, group theory multiple-choice test. We pair results from over 400 student responses with 30 follow-up interviews to illustrate the role binary operation understanding played in tasks related to a multitude of group theory concepts. We conclude by hypothesizing potential directions for the creation of a holistic binary operation understanding framework.

Key words: Binary Operation, Abstract Algebra, Student Conceptions

Binary operations are at the heart of school mathematics from early arithmetic, to high school algebra, and their generalization: abstract algebra. The prominence and familiarity of operations can lead to the belief that they are a simple concept for university-level students. We validated this conjecture through surveying a panel of introductory abstract algebra instructors. All 13 felt that the difficulty of the binary operation concept was 5 or below on a 0 to 10 scale with an average value of 2.63.

However, while students may have a strong understanding of binary operation in straight-forward contexts such as determining if a given relation is in fact a binary operation, a robust understanding is required to leverage binary operations in the contexts of building groups, differentiating between binary operations, appropriately checking properties, and dealing with unfamiliar structures. While the majority of students we surveyed could correctly determine that division is not a binary operation, understanding of binary operation seemed to contribute to incorrect responses on questions targeting understanding of group, subgroup, associative property, identities, and inverses.

In this proposal, we present results from a large-scale implementation of the Group Concept Inventory (GCI). The inventory was designed to probe conceptual understanding around fundamental topics in introductory group theory. Over 400 students, representing a multitude of institution types across the United States, responded to each question. We pair these responses with interview data to hypothesize how binary operation understanding underlies conceptions around fundamental group theory topics.

Literature Review

All group theory relies on the concept of group: a set paired with a binary operation. A binary operation is a function that maps the cartesian product of a set of elements to that set of elements. For example, addition over the integers would be a binary operation as it inputs any two integers and returns one integer. In order to understand the generalized binary operation, not only would one need to make sense of operations and their properties, but also understand binary operation as a special case of function.

The majority of research on binary operations exists within the specialized case of arithmetic operations. Slavit (1998) discussed *operation sense* in a series of stages built around familiarity with standard arithmetic operations and their relationships to other operations, properties they may possess, and their understanding independent of concrete inputs. However, this framework is built in terms of operations that are not arbitrarily defined but rather represent a standard process, such as combining groups in the case of addition. In addition to operation sense, operations have been discussed in terms of their duality as both a process and object. Gray and Tall (1994) deem the symbol associated with an operation a *procept.* An expression such as "3+2" represents both the process of adding 3 and 2, as well as the resulting sum. Similarly a function defined as f(x)=3x+4 is both a direction for how to compute an output for any input, and also an object- the function for all x-values.

As a binary operation can be any relation that is a function between a cartesian product of a set and the set itself, the generalized notion incorporates many of the complexities studied in the contexts of function. Understanding functions is challenging across grade spans (Oehrtman, Carlson, & Thompson, 2008), with their role as both processes and objects in addition to numerous representations. Notably, understanding functions (or binary operations) involves seeing function as an *action* (mapping individual inputs to outputs), *process* (a general process for mapping inputs to outputs), and *object* (that can itself be operated on such as comparing if two binary operations are the same) (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Brown DeVries, Dubinsky, and Thomas, 1997). Students have frequently proceduralized functions such as evaluating *f(x+a)* as being equal to *f(x) + a* (Carlson, 1998). Rather than coordinating what the input and output are, the students are superficially altering the function. Students have also been shown to have limitations in terms of representations, desiring an explicit rule written symbolically rather than just a correspondence of ordered pairs (Breidenbach, et al., 1992; Vinner & Dreyfus, 1989).

Two additional frameworks have been contributed in terms of undergraduate understanding of binary operation. Novotná, Stehlíková, and Hoch (2006) approached binary operation from a structure sense view dividing understanding of binary operations into four levels: *Recognise a binary operation in familiar structures; Recognise a binary operation in non-familiar structures; See elements of the set as objects to be manipulated, and understand the closure property; and See similarities and differences of the forms of defining the operations (formula, table, other).* Rather than considering stages of mental constructions in terms of process/object reification, structure sense captures abstracting from familiar objects to unfamiliar. Ehmke, Pesonen, and Haapasalo (2005) contributed an analysis in terms of procedural and conceptual understanding. They identified students as having *procedure-based* understanding of binary operation if they could match binary operations if presented in different representations. The next level is *procedure-oriented* where students could also create different representations when prompted. The highest level is *conceptual* where students could not only move between representations, but also determine if a given relation was a binary operation.

A number of studies have shown that binary operations are not a trivial topic, illustrating struggles with varying undergraduate populations including linear algebra students (Ehmke, Pesonen, & Haapasalo, 2005), abstract algebra students (Brown, et al., 1997; Dubinsky, Dautermann, Leron, & Zazkis, 1994; Hazzan, 1999), in-service and pre-service secondary teachers (Zaslavsky and Peled, 1996), and statistics students (Mevarech, 1983). Mevarech (1983) found introductory statistics students assumed that unfamiliar binary operations such as mean and variance had properties found in groups including the associative property. Zaslavsky and Peled found secondary in-service and pre-service teachers struggled to produce a binary operation that was associative, but not commutative. Binary operation related issues include defining a unary operation, and incorrectly considering repeated binary operations such as *wrongly translating* the associative property on the operation $|a+b|$ as $|a|+|b+c|=|a+b|+|c|$ rather than $||a+b|+|c||=|a+|b+c||$ or *overgeneralizing* such as considering the equality $(a*b)+c=a*(b+c)$ to determine if $(a*b)+c$ is a binary operation.

Each of these studies explored some of the complexities associated with the binary operation concept. The group theory context is often the first time that students are asked to reason about binary operations that may be unfamiliar. Furthermore, until group theory, they have likely not reasoned about the binary operation as a general concept.

Methods

These results stem from a larger project developing a concept inventory targeting conceptual understanding in introductory group theory. A 17-item instrument was developed, field-tested and refined through several rounds of validation studies (AUTHOR).The results reported here come from the final round of field-testing across the United States. Students from 33 institutions took this survey after finishing an introductory group theory portion of an undergraduate abstract algebra course. The survey was administered online. Institutions participating were geographically diverse and representative of varying levels of selectivity including 14 institutions with acceptance rates greater than 75%, 10 institutions with acceptance rates between 50-75%, and 7 institutions with acceptance rates less than 50%.

Throughout the field-testing, follow-up interviews were conducted to validate the interpretation of student responses. A total of thirty interviews were conducted including 15 with students during an open-ended round, and 15 with students completing the closed-form multiple-choice version. The students were prompted to explain their answer selection and their understanding of the relevant underlying concept.

Preliminary Results

The following results include examples of three GCI questions where understanding of binary operation appeared to influence student performance. In the first question, students are asked to define a binary operation on a set to form a group. In the second question, students determine if a given subset is a subgroup. In the third question, students evaluate if an unfamiliar operation is associative.

Students were asked to consider the set: $\{1,2,4\}$. This set was selected because it does not correlate nicely to any group students likely studied. Instead, to correctly address the prompt, students would need to recognize that a binary operation can be defined on any set with or without a symbolic rule. As can be found in Table 1, only 23% of students selected the correct response. Thirty-six percent of students responded with a familiar operation that would not meet group requirements, while the remaining students wanted the set to have additional elements in order to define a closed binary operation. This latter group represents a potential limitation in ability to construct an abstract binary operation. In follow-up interviews, a number of students explained that they tested various known operations declaring the sentiment that, "there's no operation I could think of" that would meet the requirements. In contrast, students selecting the correct response appeared to have a more sophisticated understanding of binary operation. In follow-up interviews, they explained that a binary operation can be made to meet group requirements by building an unfamiliar binary operation through leveraging alternate representations such as building a Cayley Table or defining the operation element-wise.

Table 1

Percentage of Students Selecting each Response for Defining a Group Question

Consider the set: $S = \{1, 2, 4\}$. Can an operation be defined such that S forms a group?

Response	Percentage (*n*=468)
Yes, because an operation can be defined on any three element set to form a group.	23%
Yes, multiplication mod 6.	36%
No, the set will not be closed under any operation.	18%
No, the identity element 0 would be needed.	10%
None of the above reasoning is valid	14%

Table 2 includes student response selections for the question on subgroup. This question (or a variant of it) has been used in several prior studies to illustrate student conceptions around subgroup and Lagrange's Theorem (Dubinsky, et al., 1994; Hazzan & Leron, 1996). Dubinsky et al. posited that students who identified Z3 as a subgroup of Z6 where not coordinating binary operation and set correctly - failing to see that the operation of a subgroup must be inherited from the supergroup. However, during many of the follow-up interviews conducted with students who selected the first and second option, the students articulated a notion that the subgroup's operation was "inherited." Several students explained that "it's the same operation" in Z3 and Z6, seeming to rely on a generalized version of modular addition. These students did not seem unable to recognize the need for the same binary operation, but rather did not appropriately address what it means to have the different operations. Instead of evaluating if the products of elements were the same, they instead relied on the general rule which appears to be the same type of operation.

Table 2

Percentage of Students Selecting each Response for Subgroup Question

Does the set $\{\bar{0}, \bar{1}, \bar{2}\}$ form a subgroup in $\mathbb{Z}_6$ (under modular addition)?

Response	Percentage (*n*=429)
Yes, because $\{\bar{0}, \bar{1}, \bar{2}\}$ is a subset of $\mathbb{Z}_6$.	13%
Yes, because $\mathbb{Z}_3$ is a group itself contained in $\mathbb{Z}_6$.	36%
Yes, because 3 divides 6.	6%
No, because the subset $\{\bar{0}, \bar{1}, \bar{2}\}$ is not closed.	44%

In this third question, students had to address an operation that was not associative, averaging. In relation to binary operation, there are two notable responses found in Table 3: the first where students did not feel the need to address a new operation because of its component parts being familiar associative operations, and the third option where parentheses are moved artificially. Students selecting the first choice may be superficially applying the idea of associativity being “inherited” in a new situation. Students selecting the third response fall into Zaslavsky and Peled (1996) overgeneralization category. We conjecture these students may have more fundamental issues with the binary operation procept. These students were not repeatedly operating on two elements to determine if $(a \lozenge b) \lozenge c = a \lozenge (b \lozenge c)$, but rather treating subcomponents of the binary operation as if they were three different inputs. This mimics function issues where students struggle to appropriately evaluate expressions such as $f(x+a)$. A robust understanding of binary operation requires making sense of what constitutes the input and how repeated binary operations are calculated.

Table 3

Percentage of Students Selecting each Response for Associativity Question

Consider the binary operation of averaging $\diamond$, on the set of real numbers defined below. Is this operation associative?

$$a \diamond b = \frac{1}{2}(a+b).$$

Response	Percentage (n=432)
Yes, addition (and multiplication) are associative, so $\diamond$ is also.	29%
Yes, because $\frac{1}{2}(a+b) = \frac{1}{2}(b+a)$.	22%
No, because $(\frac{1}{2}a) + b \neq \frac{1}{2}(a+b)$.	17%
No, because $\frac{a}{2} + \frac{b}{4} + \frac{c}{4} \neq \frac{a}{4} + \frac{b}{4} + \frac{c}{2}$.	31%

Discussion

The three results above illustrate some of the additional complexities associated with binary operation as found in field-testing of the GCI. Binary operation conceptions can underlie performance in a number of essential group theory tasks. Furthermore, the student responses serve as a starting ground for expansion of previous work on student conceptions of binary operation. Ehmke, Pesonen, and Haapasalo’s (2005) conceptual levels might need to be expanded where creating an unfamiliar binary operation on a given set may represent an even higher level of conceptual understanding. Novotná and Hoch (2008) identified determining if two binary operations are the same or different as the top level of binary operation understanding. This ability seemed crucial to appropriately addressing the question related to subgroup. Finally, in the associativity question, students may be more than just overgeneralizing (Zaslavsky and Peled, 1996), but have fundamental issues correctly operating. As binary operation is a special case of function, these complexities mimic many of the issues found in understanding functions for various level students. Exploring the role of binary operation can help provide insight into why students may struggle with various aspects of these algebra courses. Additional analysis of these results can hopefully build a more holistic framework of binary operation understanding,

Questions for the Audience

1. What might a comprehensive framework for student understanding of binary operation look like?
2. How might student conceptions around binary operations influence their understanding in other advanced mathematics courses?

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