Student responses to instruction in rational trigonometry

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In this paper I discuss an investigation on students' responses to lessons in Wildberger's (2005a) rational trigonometry. First I detail background information on students' struggles with trigonometry and its roots in the history of trigonometry. After detailing what rational trigonometry is and what other mathematicians think of it I describe a pre-interview, intervention, post interview experiment. In this study two students go through clinical interviews pertaining to solving triangles before and after instruction in rational trigonometry. The findings of this study show potential benefits of students studying rational trigonometry but also highlight potential detriments to the material.

Key words: [Rational Trigonometry, Undergraduate Mathematics, Interviews]

Introduction

Students struggle with trigonometry. This struggle is a contributing factor to students not pursuing studies in the STEM fields. Students struggle with trigonometry at many points during their mathematical studies. While many pedagogical changes to trigonometry instruction have been tried (Bressoud, 2010; Kendal & Stacey, 1996; Weber, 2005) little has been done looking at replacing or augmenting trigonometry instruction with a mathematical alternative.

Rational trigonometry is a system for studying triangles using different units to measure length and the separation between two lines instead of using distance and angle (Barker, 2008; Campell 2007, Franklin, 2006; Henle, 2007; Wildberger, 2005a, 2005b). The use of a different unit necessitates different formulas than traditional trigonometry. Wildberger (2005a, 2005b) claims that rational trigonometry is simpler to learn, understand, and use than its traditional counterpart. He believes this based on the formulas for rational trigonometry lacking the sine, cosine, tangent or other transcendental functions. Little if any research has been conducted looking into educational benefits of rational trigonometry.

To investigate his claims I conducted task-based interviews before and after lessons in rational trigonometry to explore the following: How do mathematics majors approaches to solving problems pertaining to triangles change after studying rational trigonometry?

Traditional Trigonometry

Trigonometry as we know and teach it causes many difficulties for students. Previous research on students' difficulties with trigonometry include studies using quantitative methods (Brown, 2005), teaching experiments (Moore, 2009, 2013; Weber, 2005, 2008), and theoretical pieces (Bressoud, 2010; Gilsdorf, Moore, 2012; Wildberger, 2005a, 2005b, 2007).

What is trigonometry?

This is a question that is rarely answered explicitly in mathematics texts (Wildberger, 2005a). One method to defining words is the etymological approach. "Tri" being the prefix for three, "gon" referring to a polygon (e.g. pentagon, hexagon etc.) and "metry" referring to measure. Putting these together yields trigonometry as the study of the measure of three sided polygons.

A second way to define a word is to look at its use throughout history. The predecessor of the sine function was developed in the second century BCE (Bressoud, 2010). This was a relationship between central angles and chords of a circle (Bressoud, 2010; Gilsdorf, 2006). Using these techniques for triangles started in the 11th century CE and was formalized as sine and cosine in the 16th century (Bressoud, 2010). Introducing students to the trigonometric functions through the use of triangles began in the 19th century (Bressoud, 2010).

A third approach to defining trigonometry is to see how the word is currently used in the literature. Looking at texts yields the following list of topics: triangles, trigonometric functions, trigonometric identities, trigonometric equations, trigonometric graphs, imaginary numbers, polar coordinates, De Moivre's theorem, McClaurin Series, integral substitutions, waves, Fourier Analysis and more (Hirsch, Fey, Hart, Schoen, & Watkins, 2009a, 2009b; Larson & Edwards 2014, Liebeck, 2005). This would lead us to defining trigonometry as the study of anything pertaining to angles, triangles, or the functions sine, cosine, and tangent.

Based on these three perspectives, trigonometry is the study of everything pertaining to the functions, which resulted from applying the study of circles, to the study of triangles. For this study I am going to focus on the mathematics of triangles.

Student difficulties with trigonometry. Many difficulties pertaining to trigonometry are well documented (e.g., Akkoc, 2008; Blackett & Tall, 1991; Bressoud, 2010; Brown, 2005; Marchi, 2012; Moore, 2009, 2012, 2013; Weber, 2005, 2008, Wildberger, 2005b). Most of the documented difficulties can be sorted into two categories: 1) difficulties pertaining to the concept of angle (Akkoc, 2008; Bressoud, 2010; Moore, 2009, 2012, 2013; Wildberger, 2005b), and 2) difficulties pertaining to the sine, cosine, and tangent functions (Bressoud, 2010; Brown, 2005; Marchi, 2012; Moore, 2012; Weber, 2005, 2008; Wildberger, 2005b).

Student difficulties with angles. Moore (2012, 2013) and Akkoc (2008) claim that student difficulties with angles stem from gaps in their teachers' understanding of angles. Bressoud (2010) attributes difficulties with angles to incompatibilities between the ratio and the unit circle approaches to understanding trigonometry. These approaches are associated with degrees and radians respectively. Students are then taught that they are interchangeable yet certain problems are to be done in terms of one and other problems in terms of the other without any justification for the decisions made (Akkoc, 2008; Bressoud, 2010). Wildberger (2005b) takes these views to an extreme by claiming that the unit itself is overly complicated and that with the exception of a few values cannot be calculated without a background in calculus.

Student difficulties with trigonometric functions. Moore (2012) attributes flawed understandings of the trigonometric functions on the volume of inconsistent definitions used for them. Brown (2005) found that students compartmentalize two different definitions for sine and cosine. These two definitions for sine and cosine are as the ordinate and abscissa respectively of points on the unit circle and as ratios of side lengths of a right triangle. Some authors have found that the meanings of the trigonometric functions are obscured by the use of the unit circle instead of the use of ratios of side lengths of right triangles (Kendal & Stacey, 1996; Markel, 1982). Markel (1982) argues that the unit circle includes angles above 180° which are unnecessary and does nothing to help students differentiate sine and cosine. Kendal (1996) found that the unit circle approach gave students more opportunities to make mistakes. However, Weber (2005) states that the unit circle was a more effective pedagogical tool than right triangles. He found that students were more likely to recognize sine and cosine as functions if taught using a unit circle approach. Students have problems viewing sine, cosine, and tangent as functions due to their non-algebraic nature and as such are unsure about how to perform algebraic operations with them (Weber, 2005). This could be due to the pedagogy straying away from beginning with the study of circles and chords (Bressoud, 2010; Gilsdorf, 2006) or it may be due to the transcendental nature of the functions (Weber, 2005; Wildberger 2005b).

Need for trigonometry. One debated topic is the importance of studying traditional trigonometry. While the importance of many mathematical topics is debated in the K-16 curriculum the inclusion and exclusion of trigonometry can be seen in multiple scenarios. Multiple groups believe that high school students are not being taught enough trigonometry and that it should be the penultimate high school course instead of calculus (Bressoud, 2012; Markel, 1982). While many college calculus courses expect a prior knowledge of trigonometry many colleges now offer variants of their calculus courses that attend to the same topics with the exception of omitting trigonometry-based problems.

Rational Trigonometry

Rational trigonometry is a reformulation of trigonometry based on replacing the units of distance and angle, with the units of quadrance and spread (Wildberger, 2005a, 2007). Quadrance is distance squared. The spread between lines l_1 and l_2 is the quadrance of $\overline{BC}$ divided by the quadrance of $\overline{AB}$ shown in Figure 1.

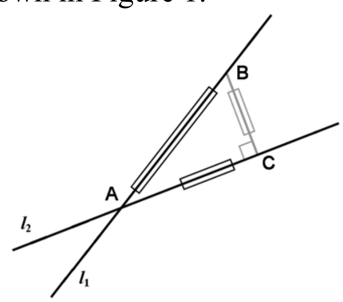

Fig. 1 The elements of the spread between two lines

Replacing the concept of angle with the concept of spread, results with the main formulas in trigonometry needing to be reformulated. The result is that the traditional trigonometry laws are replaced with the laws of rational trigonometry. They are analogous to the tradition trigonometric laws but the trigonometric functions are replaced with algebraic operations shown in Table 1 (Barker, 2008; Franklin, 2006; Henle, 2007; Wildberger, 2005a).

Table 1. Analogous Formulas in Traditional Trigonometry and Rational Trigonometry

Traditional	Rational
$c^2 = a^2 + b^2 - 2ab\cos C$	$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - S_3)$
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$	$\dfrac{S_1}{Q_1} = \dfrac{S_2}{Q_2} = \dfrac{S_3}{Q_3}$
$A + B + C = 2\pi$	$(S_1 + S_2 + S_3)^2 = 2(S_1^2 + S_2^2 + S_3^2) + 4S_1S_2S_3$

Curricular change

For something new to be adopted by the mathematics community it needs one of two things. It needs to either be able to do old tasks better than older approaches or it must be able to do new things.

Arguments in favor of rational trigonometry. Arguments in favor of rational trigonometry being simpler than traditional trigonometry are that it gets rid of the difficulties caused by the angle and the trigonometric functions by replacing them. With Rational Trigonometry, sine, cosine and tangent are no longer needed to study triangles (Barker, 2008; Franklin, 2006; Henle, 2007; Wildberger, 2005a, 2005b). Wildberger (2005a, 2005b) claims that the most complex operation needed for trigonometry becomes the square root function and that a student who has learned the quadratic formula has the prerequisite skills needed to study rational trigonometry.

Arguments against rational trigonometry. Three arguments have been made against rational trigonometry. One of these is that the units are less intuitive (Campell, 2007; Gilsdorf, 2006). Consecutive spreads of 1/4 and 1/4 combining to 3/4 is less intuitive than adding adjacent angles. Another is that many triangle problems would have irrational solutions when solved with rational trigonometry and that the irrational solutions from rational trigonometry are no more useful than the transcendental solutions from traditional trigonometry (Gilsdorf, 2006). A third is the inflexibility of the educational system (Campbell, 2007). Educational sequencing rarely changes and it pushes students to study traditional trigonometry before higher mathematics.

Questions from the arguments. The two sides of this argument bring up some interesting points in comparing the systems. Is the benefit of avoiding trigonometric functions worth a unit that is less visually intuitive? Should simpler be defined in how one uses the material or in how one learns the material? Is there any benefit to rational trigonometry when you have to study traditional trigonometry anyway?

While all of these are interesting my research question only addresses aspects of the first two. This study shows a glimpse at students working with quadrance and spread instead of the trigonometric functions. It also lets us see how two students use both trigonometric systems to address the same problems.

Mathematical research

As stated earlier there are two reasons for the mathematics community to adopt alternative mathematics. The second of these mentioned was that if it does something that has not been done before. There is a small yet existent body of literature in higher mathematics that makes use of rational trigonometry. Authors have applied the concepts of rational trigonometry to geometry (Alkhaldi, 2014; Le & Wildberger, 2013; Vinh, 2006, 2013; Wildberger, 2010), computer programming (Kosheleva, 2008), and robotics (Almeida, 2007).

Factors influencing students pursuing mathematics.

One of the factors that determines students' course taking patterns in college mathematics is their overall confidence with mathematics.

Students who expressed confidence in their mathematical abilities are more likely to take additional mathematics courses (Fennema & Sherman, 1977; Else-Quest, Hyde & Linn, 2010, Oakes, 1990). Those courses tend to be at a higher-level than the ones taken by their less confident peers (Fennema & Sherman, 1977; Else-Quest et al., 2010; Laursen, Hassi, Kogan, Hunter, & Weston, 2011; Stodolsky, Salk, & Glaessner, 1991). Typically, a loss in confidence is caused by performing lower than one's expectations (Ahmed, van der Werf, Kuyper & Minnaert, 2013). Improving students' performance in trigonometry would help their confidence and positively influence their future studies.

Students' problem solving strategies.

Students tend to use the strategies and techniques they are most recently familiar with when approaching problems (Chi, Bassok, Lewis, Reimann, & Glaser, 1989; Owen & Sweller, 1985). This explains why students might solve a quadratic by formula instead of factoring or use the law of sines when solving a right triangle. This phenomenon is stronger in weaker students who are less likely to stray from the patterns established in examples (Chi et al., 1989). Situation and context also influence how students attempt to solve problems (Moore, 2012). A student is most likely going to use the formulas they think an instructor or exam wants them to use.

As it pertains to trigonometric problems the strategies are the same in both rational and traditional trigonometry but the techniques are different. For example consider a problem where a student is given the measurements for two sides of a triangle and the vertex between them and asked for the third side. A strategy would be to use a formula that relates those four quantities. In traditional trigonometry the technique would be to use $c^2 = a^2 + b^2 - 2ab\cos C$ while in rational trigonometry the technique would be to use $(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - S_3)$.

Methodology

The comparative nature of this study influences many design decisions. Only distances are given and asked for in these tasks. To give or ask for spread or angle would inherently design the questions towards the use of a particular approach. A second outcome of this is the triangles presented in both interviews are geometrically similar. Without similarity is it possible that one interview task was inherently simpler due to the triangles used. A third result is the tasks asking for an altitude, median, and vertex bisector. These three concepts have been studied since antiquity (Heath, 1956) and as such are not dependent upon rational trigonometry for analysis.

Research Design

The inquiry approach for this study is case study design. Case study is the study of a case across a timespan (Hatch, 2002; Yin 2009). Case studies can be exploratory or explanatory in nature (Yin, 2009). For this study the cases are the two participants and the timespan is five days (the pre-interview, three days of lessons, and the post-interview).

Combining the need for a before and after and the exploratory affordances of task-based interviews (Confrey, 1981; Maher & Sigley, 2014; Schoenfeld, 2002) leads to the design of pre task based interview, lessons, post task based interview. The first interview is being used to look at strategies and techniques used by participants without a background in rational trigonometry. The lessons are used to create a background in rational trigonometry. The post interview is being used to see how a participant's behavior and/or reasoning when approaching the same task is altered after studying rational trigonometry.

Three video lessons on rational trigonometry were given to the participants. I designed these lessons to give familiarity with the units and formulas for rational trigonometry. The first lesson was focused on the units. The second lesson focused on the formulas $(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - S_3)$ (the cross law) and $\frac{S_1}{Q_1} = \frac{S_2}{Q_2} = \frac{S_3}{Q_3}$ (the spread law). The third lesson focused on $(S_1 + S_2 + S_3)^2 = 2(S_1^2 + S_2^2 + S_3^2) + 4S_1S_2S_3$ (the triple spread law). Each lesson was accompanied with a worksheet that acted as practice for the participant, additional data for myself, and verification that they watched the videos.

The first lesson focused on the units of quadrance and spread. Quadrance was described as distance squared and spread was first defined geometrically. After that I detailed arithmetical properties of spread and showed examples of how to calculate the spread for lines given in both slope-intercept and standard forms. Spread bisection was also shown. The second lesson focused on the spread and cross laws. The lesson was an example of solving a triangle knowing the quadrances of two sides and the spread between them. It started with the cross law being used to find the missing side and was followed by using the spread law to find the two remaining spreads. The last lesson focused on the triple spread law. The lesson was an example of using the triple spread formula to find the third spread in a triangle if only two spreads were known.

Participants

Due to the comparative nature of this study, participants with a strong background in mathematics in general, trigonometry in particular, and with no background in rational trigonometry were recruited. To ensure this, mathematics students with a 4.0 in their first year mathematics courses including Euclidian trigonometry were chosen.

Data Collection

Data was collected through a pre-interview, three worksheets and a post-interview.

Interviews. Task based interviews were used to gather information about the participants approaches to solving problems pertaining to triangles. The two interviews were audio recorded and occurred four days apart. Between the pre and post interviews the participants watched all three lessons and completed all three worksheets. Participants were supplied with pencil, paper, and a selection of traditional trigonometric formulas. During the second interview they were also given the rational trigonometry formulas from the lessons. From the interviews both their spoken word and written work were collected.

The three tasks chosen for the interviews were chosen to have no inherent bias towards traditional or rational trigonometry. The first task was to find the length of an altitude of a triangle. This task is commonly shown in the high school curriculum and is often done with and without the use of the sine, cosine, and tangent functions (Keenan & Gantert, 1989; Hirsch, et al., 2009b). The second task was to find the length of a median of a triangle. The last task was to find the length of a vertex bisector.

Worksheets. The primary purpose of the worksheets was to ensure that the participants watched the videos. The work was analyzed with respect to the findings from the interviews for triangulation purposes. All three worksheets were collected at the second interview.

Data Analysis

I began my data analysis by transcribing the interviews. At this point I was already making decisions about what data had the potential to show interesting findings. After this my next step was coding the data. That data was separated and regrouped for organizational purposes (Creswell, 2014; Maxwell, 2013; Saldana, 2009; Seidman, 2012). My coding efforts were focused on the written work and verbal statements given during the interviews. Once this was done I focused on the findings that were most abundant and different between both interviews. The strongest examples are highlighted here.

Findings

After my analysis three themes emerged. These themes were strategies, numerical properties of triangles, and confidence. The strategies used involved the Pythagorean theorem and the relationships represented by the laws of sine and cosine and the spread and cross laws. Numerical properties were that distances must be positive, the triangle inequality, and that the longest sides of a triangle are across from the largest angles / spreads.

Maureen

Maureen is a mathematics major with the goal of becoming a high school mathematics teacher. Her undergraduate course on trigonometry ended four months before the study.

Strategies. Maureen started the pre-interview using the Pythagorean theorem in an attempt to find the value of an altitude. After multiple iterations gave her more unknowns than equations or values that did not make sense to her she abandoned this strategy. Her next attempt was to use the Law of Cosines to find one of the angles. Her goal was to use that angle in the Law of sines to find the altitude. Once she found $\cos\theta = \frac{13}{14}$ she abandoned that approach as well.

During the second interview Maureen used the cross and spread laws in the manner she intended to use the Laws of cosines and sines in the first interview. In this attempt she successfully used both formulas. Though her use of the spread law gave her the quadrance of the altitude she did not turn that value into a length as the question was asking for. When questioned she said that the answer she gave was the length of the altitude.

N**umerical properties.** During the pre-interview Maureen made ample use of numerical properties of triangles. In particular she made use of the fact that side lengths cannot be negative and she made use of the triangle inequality. She used these to check her computational results.

The triangle inequality was also used to determine ranges for the answers to the interview tasks. Since she did not compute an angle there was no opportunity to observe if she would have used that the longest side is opposite the largest angle.

In the post interview there was no use of the triangle inequality. This could have been used to alert her to not having the right answer in the first task. She did however use the property that the largest spread has to be across from the largest side of a triangle.

Confidence. Maureen's confidence in approaching these tasks appeared to increase after the lessons in rational trigonometry. In the first interview she spent a lot of time staring at the tasks without performing any calculations. After a particularly long silence she said:

> As much as I hate to admit that I can not remember how to solve for altitude, I'm just going to spend 20 minutes staring at this, because I'm not liking what I'm getting. I feel very bad saying that and admitting that, but it's not gonna happen.

In the second interview the gaps in work and expressions of frustration lessened. After the interview she gave the following two statements: "That was really really cool the whole quadrance and [spread]" and "if I had more time to practice I think I could have gotten all 3." These statements point towards a higher confidence level using rational trigonometry.

Tom

Tom is a mathematics major aiming towards graduate studies in applied mathematics. He took his trigonometry course approximately three years before the study.

Strategies. In the first interview Tom's strategy was to solve for anything he could find in hopes that he would come up with pieces he needed to solve the tasks. When he found $\cos\theta = \frac{13}{14}$ he used that value in another Cosine Law equation in order to solve one of the tasks.

In the second interview his strategies were nearly identical. The biggest change between the two interviews was he was using the rational trigonometry formulas instead of the traditional trigonometry formulas.

Numerical relationships. There was no evidence in either interview that Tom used the numerical properties listed above. He submitted answers to all three tasks and he could have but did not find two of them to be impossible due to the triangle inequality. In both interviews he was confident in his strategies (which would have worked) and his computations (which contained errors).

Confidence. Tom showed no notable change in confidence.

Discussion

Based on the findings I believe it is safe to say there may be some benefits to students studying rational trigonometry. The strongest evidence for benefits come from Maureen's case. Maureen falls into the category of students who are weaker with their algebraic manipulation of functions, which hindered her mastery of trigonometry (Weber, 2005). She seemed to increase in confidence after studying rational trigonometry and appeared more capable of solving problems when using the rational trigonometry formulas. Tom showed a strong mastery of the algebra of functions and little change in performance using the rational formulas. This may point to potential benefits being more likely for students with a weaker skill set pertaining to functions.

Potential weaknesses also need to be mentioned. Maureen did not apply the numerical properties that she showed earlier evidence of using. She also at one point equated quadrance and distance. Quadrance being less intuitive than distance (Campbell, 2007; Gilsdorf 2006) is likely a contributing factor of this.

In conclusion there is more to research here. While benefits may exist it is possible that they are outweighed by the costs.

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