Effect of emphasizing a dynamic perspective on the formal definition of limit

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ABSTRACT. We attempt to determine the efficacy of using an alternate, equivalent formulation of the formal definition of the limit in a first-year university calculus course in aiding the understanding of the definition and of alleviating the development of common misconceptions concerning the limit.

Introduction

Students face many difficulties in learning calculus, and in learning the concept of limit in particular (see Tall, 1993; Williams, 1991, for example). One difficulty that is commonly experienced by students is in resolving their intuitive, dynamic, step-by-step view of the limit of a function with the static, continuous point of view required for understanding the formal ε-δ definition. This difficulty is exacerbated by the fact that the language that both textbooks and instructors use in discussing limits is rife with dynamic terms like "approaching," and dynamic notation like the $\rightarrow$ symbol (see Monaghan, 1991).

In making formal definitions, mathematicians usually have foremost in mind utility for theoretical development, with a generous helping of an appreciation for elegance and conciseness. Such goals can be at odds with the tasks of teaching and learning. The formal definition of limit in calculus is a notoriously difficult topic for introductory calculus students to grasp, let alone develop deep understanding. It captures the notion of limit in a succinct and mathematically rigorous manner, but for neophytes the connection between informal idea and formal definition is often lost in a soup of Greek letters, mathematical symbols, and nested logic. Given that it is often possible to formulate a formal definition in multiple equivalent ways, might it not be preferable for mathematics educators, at such an introductory level, to choose a definition most conducive to teaching and learning rather than a definition conducive to doing mathematics in the style of professional mathematicians? In this study, we attempt to investigate whether a definition using convergence of sequences might meet this criterion. To begin to study the effectiveness of such a definition, we study its effect on the development of common misconceptions concerning limits.

Methodology

In the fall term of 2014, pre- and post-tests were administered to students across four sections of two first-year introductory calculus courses, and results from 134 participants were retained. The first author was the instructor for two sections (henceforth referred to as 'Section A1' and 'Section A2') of 'Course A,' a standard introductory calculus course attended by students both with and without prior calculus exposure at the high school level. The second author was the instructor for a third section (henceforth 'Section A3') of Course A and for the sole section (henceforth 'Section B') of 'Course B,' a slightly more challenging introductory calculus course with prior calculus exposure at the high school level as a prerequisite. In Course A, the usual ε-δ formal definition of limit was used, while in Course B, an equivalent definition using convergence of sequences

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was introduced. Students in Course B did not have a preceding in-depth unit on sequences and convergence, relying only on a brief introduction to sequences sufficient for their use in the study of the limit. The relevant definitions are given below.

Definition. *A sequence $\{x_n\}$ **approaches** c (or **converges to** c) if for every possible error bound E, no matter how small, all but a finite number of terms lie within $c \pm E$.*

Definition. *We say that **the limit of** f **as** x **approaches** c **is** L, and write $\lim_{x\to c} f(x) = L$, when* every possible *input sequence $\{x_n\}$ with $x_n \to c$ (but $x_n \neq c$) produces an output sequence $\{y_n\}$ with $y_n \to L$.*

In the definition of convergence of sequences above, we chose a definition appropriate for an introductory calculus course, rather than the more technical ε-N version typical of more advanced analysis courses. Also, the notation $c \pm E$ represents the interval $(c-E, c+E)$, and was used to be consistent with the notation of error bounds in other scientific disciplines.

Table 1. Concepts and misconceptions evaluated on the pre/post-tests.

Question	Concept/Misconception
1	Limit versus value of a function.
2	A function's values must approach but never equal the limiting value.
3	A finite table of values is enough to determine a limit with certainty.
4	The values of a function always monotonically approach the limiting value.
5	Input-output order reversal/confusion in the formal definition.
6	Negation of the formal definition.

In each test, students were given six multiple choice questions designed to test common misconceptions about the limit. The pre-test was administered shortly after the concept of limit was introduced. The formal definition of the limit and all discussion regarding formal justification of limit properties were then delayed until the end of the unit on limits. The post-test was administered some time after discussion of the formal definition of the limit, to give the definition and its relationships to the idea and calculation of limits time to sink in. The post-test was identical to the pre-test, with the exceptions that the order of the questions, the order of the possible answers for each question, and some of the numbers involved in the questions were changed. The students did not know beforehand that the questions on the post-test would be essentially the same as the questions on the pre-test, and students were not given answers to either test until both tests were completed. The concept or misconception that each question was designed to measure is listed in Table 1.

Results

Question 1 asked students whether it is possible for $\lim_{x\to c} f(x) = f(c)$ to be true for a given function f. From experience with elementary functions, students often are under the misconception that this equality is *always* true. Or, they believe that the equality cannot hold because a function cannot actually "reach" its limit. In Table 2, we summarize the student responses to this question. In the table headings for this table and all subsequent tables, the symbols ✓and ✗ are used as shorthand for "correct" and "incorrect," respectively, and an arrow indicates a change in response between the pre- and post-tests.

Question 2 asked students about the relationship between a known limiting value L of a function f, the value $f(c)$ (if any) at the point of interest c, and the values $f(x)$ at points x near c. Students

Table 2. Student responses on question 1.

Section	N	Pre ✓	Post ✓	✗→✓	✓→✗
A1	37	35%	76%	49%	8%
A2	29	38%	83%	55%	10%
A3	35	57%	66%	23%	14%
A (all)	101	44%	74%	42%	11%
B	33	70%	85%	21%	6%

often believe that the values at neighbourhood points must be close to L, but should not equal either L or $f(c)$. Student responses to Question 2 are summarized in Table 3.

Table 3. Student responses on question 2.

Section	N	Pre ✓	Post ✓	✗→✓	✓→✗
A1	37	8%	30%	27%	5%
A2	29	3%	7%	7%	3%
A3	35	6%	11%	9%	3%
A (all)	101	6%	17%	15%	4%
B	33	27%	15%	3%	15%

In Question 3, students were given a finite table of values for a function at points near a point of interest c, both to the left and to the right, where the values appeared to be approaching a specific common value from both sides of c. They were then asked about possible inferences that could be made about the limit of the function at c. The insufficiency of a finite table of values in establishing a limit with certainty is a fundamental issue addressed by the formal definition of the limit. Table 4 summarizes the student responses to this question.

Table 4. Student responses on question 3.

Section	N	Pre ✓	Post ✓	✗→✓	✓→✗
A1	37	16%	14%	11%	14%
A2	29	38%	28%	0%	10%
A3	35	17%	34%	26%	9%
A (all)	101	23%	25%	13%	11%
B	33	30%	24%	3%	9%

In Question 4, students were given a limiting value and a nearby value of a function, and were asked about possible inferences that could be made about the value of the function at a point in between the two given values. Students often conceptualize a limit as always being monotonically approached by the values of the function nearby, and this question aimed to expose such misconceptions. The student responses are summarized in Table 5.

Questions 5 and 6 aimed to probe students' understanding of the formal definition in plain language. In Question 5, students were required to choose the phrase that correctly justified a limit, where the incorrect phrases contained various forms of input-output reversal. Because students are used to the input-output order of function evaluation, they often struggle with the process of working in the reverse order, starting with an arbitrary interval of y-values around the limiting value and working backwards to an appropriate interval of x-values around the point of interest.

Table 5. Student responses on question 4.

Section	**N**	**Pre ✓**	**Post ✓**	**✗→✓**	**✓→✗**
A1	37	14%	11%	11%	14%
A2	29	7%	17%	14%	3%
A3	35	3%	11%	9%	0%
A (all)	101	8%	13%	11%	6%
B	33	18%	12%	6%	12%

In Question 6, students were required to choose the phrase that correctly justified why a proposed value could *not* be the limit of an example function. Because of the nested logic involved in the formal definition of the limit, students struggle with understanding the negation of the definition. It was hoped that the convergence version of the formal definition would aid with this task, as in this version the nested logic broken out into the separate definitions of convergence and limit via convergence.

Responses to Questions 5 and 6 are summarized in Tables 6 and 7.

Table 6. Student responses on question 5.

Section	**N**	**Pre ✓**	**Post ✓**	**✗→✓**	**✓→✗**
A1	37	14%	38%	30%	5%
A2	29	28%	38%	21%	10%
A3	35	14%	17%	14%	11%
A (all)	101	18%	31%	22%	9%
B	33	15%	21%	18%	12%

Table 7. Student responses on question 6.

Section	**N**	**Pre ✓**	**Post ✓**	**✗→✓**	**✓→✗**
A1	37	22%	32%	22%	11%
A2	29	41%	14%	7%	34%
A3	35	23%	11%	11%	23%
A (all)	101	28%	20%	14%	22%
B	33	24%	30%	27%	21%

Overall, the initial results from the pre- and post-tests are disappointing: on most questions, students who learned the formal definition via sequences (Section B) performed more poorly on the post-test than on the pre-test. Question 6 in particular was one on which it was expected that unchaining the nested logic of the formal definition into separate definitions would be of benefit to the students. However, while the percentage of students in Section B answering this question correctly increased from the pre-test to the post-test (and the percentage in the other sections actually decreased), we see that almost as many students in Section B moved from correct thinking to incorrect between tests as did vice versa.

Conclusion

In this study, we attempted to determine the efficacy of using an alternate, equivalent formulation of the formal definition of the limit in aiding the understanding of the definition and of alleviating the development of common misconceptions concerning the limit. On both counts, the alternate definition did not seem to produce any improvement in student performance over the traditional definition, and perhaps could be construed to actually have produced a decrease in student performance.

These results might be explained by factors other than the definitions used. First, the instructors of the courses involved in the study have many years experience teaching first-year calculus using the traditional formal definition, and with that experience comes the knowledge of typical student difficulties with the material and strategies to mitigate those difficulties. But for the instructor of the experimental Section B, teaching a formal definition in terms of sequences was a new experience. Second, sequences were only introduced in Section B at the end of unit on limits, to facilitate the discussion of the formal definition. The sequence version of the formal definition might be more relevant and attractive to the students if the entire unit on limits was infused with sequences.

On the other hand, it may be the case that the formal definition of the limit via sequences, while unchaining the nested logic of the traditional definition into two separate definitions, has merely pushed the conflict between informal, dynamic conception and formal, static definition to the definition of convergence of sequences. Or, it could be that briefly introducing sequences merely introduces a significant, preliminary learning barrier, beyond which students were unable to progress to the learning of the formal definition of the limit.

Some questions, the discussion of which would help further this research, follow.

- Would it be worthwhile to reverse the set up of the experiment and have students in Course A (who mostly have not previously studied calculus) exposed to a sequences version of limits?
- Should the pre- and post-tests focus more on understanding of the formal definition itself?
- In the experimental version, formal understanding of limit is dependent on formal understanding of convergence of sequences. Should understanding of sequences be simultaneously tested, to be correlated with understanding of limits?

References

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