An example of a linguistic obstacle to proof construction: Dori and the hidden double negative

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This paper considers the difficulty that university students' may have when unpacking an informally worded theorem statement into its formal equivalent in order to understand its logical structure, and hence, construct a proof. This situation is illustrated with the case of Dori who encountered just such a difficulty with a hidden double negative. She was taking a transition-to-proof course that began by having students first prove formally worded "if-then" theorem statements that enabled them to construct proof frameworks, and thereby, make initial progress on constructing proofs. But later, students were presented with some informally worded theorem statements to prove. We go on to consider the question of when, and how, to enculturate students into the often informal way that theorem statements are normally written, while still enabling them to progress in their proof construction abilities.

Key words: Transition-to-proof, Proof Construction, Informally Worded Theorem Statements, Proof Framework, Unpacking

This paper considers linguistic obstacles[1] that university students often have when unpacking informally worded mathematical statements into their formal equivalents. This can become especially apparent when students are attempting to prove such statements. We illustrate this with an example from Dori, who was taking a transition-to-proof course that began by having students construct proofs for formally worded "if, then" theorem statements. Early on, she was introduced to the idea of constructing proof frameworks (Selden, Benkhalti, & Selden, 2014; Selden & Selden, 1995) and was successful. Later, she encountered difficulty when attempting to interpret and prove an informally worded statement with a hidden double negative. First we will introduce our theoretical perspective and the idea of proof frameworks.

Theoretical Perspective

We adopt the theoretical perspective as described in Selden and Selden (2015); that is, we consider a proof construction to be a sequence of mental or physical actions, some of which do not appear in the final written proof text. Each action is driven by a situation in the partly completed proof construction and its interpretation (Selden, McKee, & Selden, 2010). For example, suppose that in a partly completed proof, there is an "or" in the hypothesis of a statement yet to be proved: *If A or B, then C*. Here, the situation is having to prove this statement. The interpretation is realizing that *C* can be proved by cases. The action is constructing two independent sub-proofs; one in which one supposes *A* and proves *C*, the other in which one supposes *B* and proves *C*.

We also note that a proof can be divided into a formal-rhetorical part and a problem-centered part. The *formal-rhetorical* part is the part of a proof that depends only on unpacking and using the logical structure of the statement of the theorem, associated definitions, and earlier results. In general, this part does not depend on a deep understanding of, or intuition about, the concepts involved or on genuine problem solving in the sense of Schoenfeld (1985, p. 74). Instead it depends on a kind of "technical skill". The remaining part of a proof has been called the *problem-centered* part. It is the part that *does* depend on genuine problem solving, intuition, heuristics,

[1] The idea of linguistic obstacles to learning mathematics is not new to mathematics education research (Boero, Douek, & Ferrari, 2002; Ferrari, 1999). However, to our knowledge, no one has previously discussed hidden double negatives.

and a deeper understanding of the concepts involved (Selden & Selden, 2013).

One might suppose that the problem-centered part of a proof is the most important part, and as students make progress in their proof construction ability, this may be true. However, for students of a one-semester transition-to-proof course, constructing the formal-rhetorical part of a proof can be non-trivial, yet easier to learn than the construction of the problem-centered part. Furthermore, first writing some of the formal-rhetorical part of a proof is often helpful for constructing the problem-centered part of the proof because the formal-rhetorical part exposes the "main problem" to be solved (Selden & Selden, 2009).

Proof Frameworks

A major feature that can help one write the formal-rhetorical part of a proof is a *proof framework*, of which there are several kinds, and in most cases, both a first-level and a second-level framework. For example, given a theorem of the form "For all real numbers x, if $P(x)$ then $Q(x)$", a first-level proof framework would be "Let x be a real number. Suppose $P(x)$. … Therefore $Q(x)$," with the remainder of the proof ultimately replacing the ellipsis. A second-level framework can often be obtained by "unpacking" the meaning of $Q(x)$ and putting its (second-level) framework between the lines already written for the first-level framework. Thus, the proof would "grow" from both ends toward the middle, instead of being written from the top down. In case there are subproofs, these can be handled in a similar way. A more detailed explanation with examples can be found in Selden, Benkhalkti, and Selden (2014). A proof need not show evidence of a proof framework to be correct. However, we have observed that use of proof frameworks tends to help novice university mathematics students write correct, well-organized, and easy-to-read proofs (McKee, Savic, Selden, & Selden, 2010).

The Formal-Informal Distinction and Linguistic Obstacles

An *informal statement* is one that departs from the most common natural language version of predicate and propositional calculus or fails to name variables. For example, the statement, "differentiable functions are continuous," is informal because a universal quantifier is understood by convention, but is not explicitly indicated, because the variables are not named, and because it departs from the familiar "if-then" expression of the conditional. Such statements are commonplace in everyday mathematical conversations, lectures, and books. They are not ambiguous or ill-formed because widely understood, but rarely articulated, conventions permit their precise interpretation by mathematicians, and less reliably, by students. In our experience, mathematicians, including those with no formal training in symbolic logic, move easily between informal statements and their equivalent more formal versions.

We conjecture that an informal version of a theorem will often be more memorable, that is, more easily remembered and brought to mind, but also be more difficult to prove, and also, given a proof, be more difficult to validate, than a formal version. This suggests the question: Can undergraduates who have taken a transition-to-proof course reliably unpack an informally stated theorem into its formal version? Our earlier paper on students' unpacking of the logic of statements (Selden & Selden, 1995) indicates that the answer to this question is often no. Because validation is difficult to observe directly, data were collected to determine whether the participants could reliably unpack, rather than validate, informally written statements, many from calculus, into their formal equivalent versions. (See Selden and Selden, 1995, for details.)

Because the inability to unpack an informally written theorem statement into a formal version can often prevent a student from constructing a proof, we think that the informal way that a theorem is stated can be a *linguistic obstacle.* Such an obstacle need not be a mistake or misconception (i.e., believing something that is false). Indeed, the obstacles mentioned in the Selden and Selden (1995) paper are related to difficulties with unpacking the logic of informally worded mathematical statements. In what follows, we extend this work and examine a linguistic obstacle of a somewhat different kind.

The Case of Dori and the Hidden Double Negative

What happens when a student is confronted with a hidden double negative in a theorem statement and wants to construct a proof? We report, using our field notes and photos, on a mathematics graduate student, Dori, who in a tutoring session at the end of an inquiry-based transition-to-proof course, was confronted with the task of proving: *A group has no proper left ideals*. (This was in a semigroup setting, in which L is a left ideal of a semigroup S if $L \subseteq S$ and $SL \subseteq L$.) Dori had already experienced proving theorems on sets, functions, real analysis, and abstract algebra (semigroups). She had available to her the course notes, with all previous definitions and theorems. In the same tutoring session, she had just taken 40 minutes to prove, with some difficulty and backtracking, that group inverses are unique. Specifically, she had just proved: *Let G be a group with identity* 1. *If* g, g', $g'' \in G$ *with* $gg' = g'g = 1$ *and* $gg'' = g''g = 1$, *then* $g' = g''$.

Dori, who was working at three seminar room blackboards, next began to prove a theorem about left (semigroup) ideals by writing the theorem statement on the middle board . She wrote: *A group has no proper left ideals*. Dori then looked up various definitions, such as that of left ideal and proper, in the course notes. We then talked with her about what “not proper” means, after which she wrote $GI \neq I$ and $GI \subset I$ on the right board and suggested doing a proof by contradiction. We were surprised at this suggestion, and now speculate this might have been because of the word “no” in the theorem statement. At the time, however, realizing that this would not be a productive approach, we suggested that Dori write a proof framework as she had been accustomed to doing in the past. She continued writing, below the theorem statement on the middle board:

Suppose G is a group and I is a left ideal of G.
…
Then G=I.
Therefore, I is NOT a proper left ideal of G.

We then suggested that Dori write in her scratch work the properties of a group and of a left ideal of a semigroup. She wrote these additional observations correctly on the right board. These included noting that G has an identity and inverses, that I being a left ideal means that $GI = \{gi \mid g \in G, i \in I\}$, $GI \subseteq I$, and $I \neq \varnothing$. Dori also noted the existence of the identity element, $1 \in G$ and that there is an $i \in I$ and so $i \in G$. In addition, Dori drew an appropriate diagram of the situation, with one circle labeled I contained in a larger circle labeled G, with $i \in I$ and 1in the space between the two circles. (See Figure 1.)

The emphasis, in Dori’s scratch work, on what it means for I to be a proper ideal may not have been helpful, as she, according to her proof framework, was trying to show that I was *not* proper, namely, the negation. It is often difficult for university students to form proper mathematical negations; instead, they often formulate the opposite, as they would in everyday life (Antonini, 2001). Somehow, Dori did *not* note, at this point, that in order to show that $G=I$ (the penultimate line of her proof framework), all she needed to show was $G \subseteq I$. One can speculate on why this might have been.

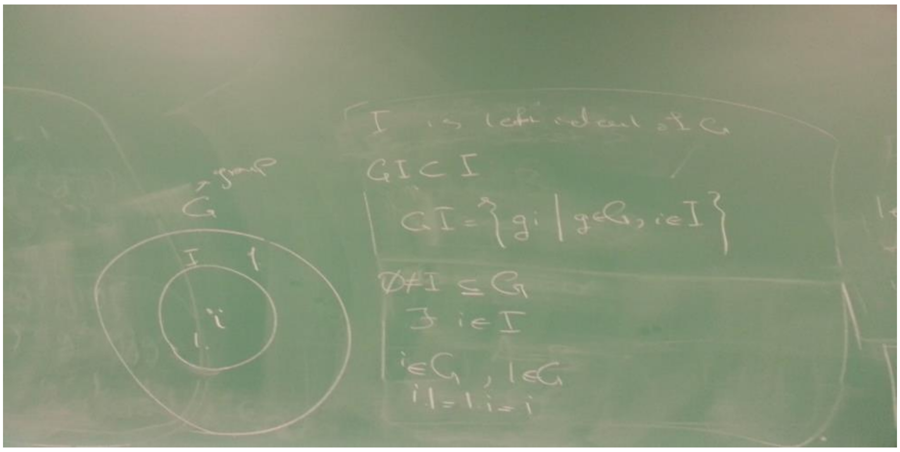

Figure 1. Some of Dori's scratch work with the diagram.

Difficulties Inherent in Converting the Theorem Statement to its Formal Version

As Dori was working diligently on her scratch work, it appeared to one of us that the informal wording of the theorem statement might be causing Dori difficulty. So, while Dori continued her scratch work, this one of us decided to try to translate the theorem into "if-then" format, judging that it might be easier to comprehend. It became clear that there were two negations involved in the phrase "no proper". The first was contained in the word "no". The second was hidden within the word "proper", which means that the ideal, I, is a proper subset of G, namely, that $I \neq G$. Thus, there is a double negation in the statement of the theorem. Having noted this, we went on to use this observation to write the theorem statement in a positive "if-then" way as, *If I is a left ideal of G, then I=G,* on the left blackboard. We went over this version of the theorem statement with Dori. The positive "if-then" formulation of the theorem has the following apparent advantages: (1) The notation has been introduced. (2) It is in the formal "if-then" form, from which a proof framework can be written in a straightforward way. (3) It does not have a hidden double negation, but rather is entirely positive and straightforward.

Dori had had no trouble introducing the notation. Perhaps this was because of the theorem on inverses that she had proved earlier that day; it already contained the notation G for a group and 1 for the identity element. With encouragement from us, towards the beginning of her proof attempt, Dori had written a proof framework, introducing the letters G for the group and I for a left ideal of G, and scrolling to the bottom, had written $G=I$ in the penultimate line and had concluded in the final line that I is not a proper left ideal of G, as well as having produced some scratch work (Figure 1). After discussing with her the positively worded version of the statement, namely, *If I is a left ideal of G, then I=G,* we suggested that she "Suppose $1 \in I$" to see what happens. Dori wrote "Let $1 \in I$" and also, "Let $g \in G$, $i \in I$, so $gi \in I$. Let $i=1$, so $g \cdot 1 = g \in I$." This essentially completes the argument that $G \subseteq I$, and hence, proves the theorem. From start to finish, this entire proving episode took 45 minutes.

To recapitulate, to prove the theorem, one observes, as Dori had, that ideals are non-empty, so there is an $i \in I$, that $i^{-1} \in G$, and hence, $i^{-1} \cdot i \in I$ because I is a left ideal. That means $1 = i^{-1} \cdot i \in I$, But if $1 \in I$, then $g = g \cdot 1 \in I$ for any $g \in G$. So $G \subseteq I$.

The Hidden Double Negation

Did the presence of a hidden double negation in the informal version of the theorem statement cause Dori difficulty? We cannot say for sure. However, it seems quite clear that the informal version of the statement, like many such informal versions, while definitely memorable, is difficult for students to unpack into its formal (positive) version. It is well-known to cognitive psychologists that negations are hard to decode and understand. Pinker (2014) reasoned as follows:

> The cognitive difference between believing that a proposition is true (which require no work beyond understanding it) and believing that it is false (which requires adding and remembering a mental tag), has enormous implications for a writer [and a reader]. The most obvious is that a negative statement like *The king is not dead* is harder on the reader than an affirmative one like *The king is alive.* Every negation requires mental homework, and when a sentence contains many of them the reader can be overwhelmed. Even worse, a sentence can have more negations that you think it does. Not all negation words begin with *n*; many have the concept of negation tucked inside them, such as *few, little, least, seldom, though, rarely, instead, doubt, deny, refute, avoid,* and *ignore.* The use of multiple negations in a sentence … is arduous at best and bewildering at worst …" (pp. 172-173).

The word "proper" in the above informally worded theorem statement has the concept of a double negation "tucked" inside it, and according to Pinker, would be arduous and bewildering.

Transitioning Students from Proving Formally Stated Theorems to More Informally Stated Theorems

When we write our course notes, we begin by including all notation and write the statements of theorems in "if-then" format, which allows students to write at least a first-level proof framework without difficulty. In addition, if they can "unpack" the conclusion (the final line of their emerging proof), they can produce a second-level proof framework. This goes a long way to exposing the real mathematical problem to be solved in order to construct the rest of the proof. Eventually, during our course, we begin to transition students to less formal ways of stating theorems, by for example, having them prove*: Every semigroup can have at most one identity element and at most one zero element.* Here the difficulty is not in introducing notation, but in deciding what "at most" means and how to structure a proof of it (namely, by assuming there are identity elements *e* and *f* and using the definition of identity in a clever way).

Perhaps a better progression would be to have students first prove a number of "if, then" theorems with all notation included, and then to have them introduce the notation and reformulate an "easy" informal statement into its formal version. For example, we might ask students to prove: *The composition of two* 1-1 *real functions is* 1-1, omitting the names of the two functions. Here it is relatively easy to introduce notation, *f* and *g*, for the two functions, and to put the statement in "if-then" form." In addition, composition has been defined in the course notes so there are no decisions to make on how to structure a proof, provided students can unpack the definition of composition. We feel such a rearranged course design would help increase student success and the early building of a sense of self-efficacy (Bandura, 1994, 1995), while gradually transitioning students to more informally worded theorem statements that are more difficult to unpack.

We anticipate that further theoretical and linguistic comments and conjectures will be included in the presentation.

Discussion Questions

1. What sorts of problems do students have in unpacking informally worded theorem statements, other than: (1) Suitable notation has not been included and has to be introduced. (2) It is not in "if-then" format, so it is not clear how to structure a proof (i.e., it is unclear how to construct a proof framework) . (3) Lack of positive phrasing (e.g., hidden double negations)?
2. What are some possible progressions that would help undergraduate mathematics students transition to being able to interpret informally stated mathematical theorems into their formal equivalents in order to construct proofs of them? How would one research their effectiveness?
3. Would it be an interesting research project to examine a variety of undergraduate textbooks to determine how many theorems are stated in an informal, possibly confusing, way?

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