Effect of teacher prompts on student proof construction

Margaret Morrow — SUNY—Plattsburgh
Mary Shepherd — Northwest Missouri State University

Many students have difficulty learning to construct mathematical proofs. In an upper level mathematics course using inquiry based methods, while this is some research on the types of verbal discourse in these courses, there is little, if any, research on teachers' written comments on students' work. This paper presents some very preliminary results from ongoing analysis from Morrow's written prompts on students' rough drafts of proofs for an Abstract Algebra course. The teacher prompts will initially be analyzed through a framework proposed by Blanton & Stylianou (2014) for verbal discourse and the framework will be modified in the course of the analysis. Can we understand if a type of prompt is "better" in some sense in getting students to reflect on their work and refine their proofs? It is anticipated that teacher prompts in the form of transactive questions are more effective in helping students construct proofs.

Key words: Abstract Algebra, transactive questions, mathematical proof, Inquiry-Based Learning

Overview

Many students, who are quite successful at lower level undergraduate mathematics courses where calculations and applications are stressed, have difficulty as they learn to construct mathematical proofs, the focus of upper level mathematics courses (Harel & Sowder, 1998; Weber, 2001; Raman, 2003). Speer, et al. (2010) call for more research in the practice of mathematics teaching at the undergraduate level. As active learning approaches to teaching (as opposed to straight lecturing methods) become more prevalent in undergraduate mathematics classes, teacher skills of listening to students, and responding to their ideas becomes ever more important. There has been a little research on the verbal discourse that occurs in these classrooms that emphasize active learning and inquiry based learning methods (Blanton & Stylianou, 2014; Johnson, 2013; Remillard, 2014). Yet, teachers also interact with students when they (the teachers) comment on students' written work. At least in both authors' classrooms, we comment on student work and we expect students to read our comments and revise their work based on our comments. But little research has been found in mathematics education research (or physics or engineering education) that deals specially with the types of comments mathematics teachers make on written work, or the effect of these comments on revisions of student work. There is a body of research in the Rhetoric and Composition discipline on feedback but its applicability to the writing of proofs seems limited.

One of the active teaching methods used at the collegiate level is Inquiry Based Learning (IBL). In a common form of this, students are given a "list" of definitions and theorems, maybe some problems, that they work through and present to the class. In the Spring 2015, Morrow, in an Abstract Algebra class, had students prepare rough drafts of the proofs they would present the day before the proofs were to be presented in class. These drafts were hand written and submitted directly to Morrow. She had about four hours to provide short written prompts (comments) to students on their rough draft proofs. Copies of the drafts with comments were made and retained by Morrow as the initial artifact of the research. The drafts with comments were then given back to the students. It was the intention that the students would use these prompts to refine their proofs before class meeting. These second drafts of proofs were maintained in a portfolio by the students and collected at the end of the semester. Copies were made of the proofs corresponding to the rough drafts received. In reviewing the proofs in the portfolios, it seemed some of the prompts proved effective, some not so much.

Literature Review and Theoretical Framework

Mathematicians understand and believe that constructing proofs is a creative process that can involve imagery, heuristics and intuition (Raman, 2003). Mathematical creativity has been described as the process that results in insightful solutions to a given problem (Sriraman, 2004). IBL methods attempt to get students to practice this creative craft of mathematics. In an IBL approach to a classroom, teachers must be “active participants in establishing the mathematical path of the classroom community while at the same time allowing students to retain ownership of the mathematics." (Johnson, 2013) This ownership is in part in the form of creating their own proofs of the various statements, an often messy process. The IBL process, as implemented in Morrow’s class, requires students to work alone without consulting tutors, fellow students or other resources. It not only calls for students to present their proofs but also calls for the other students to evaluate and validate (or not) the presented proofs. This is often done in whole class discussions.

Blanton & Stylianou (2014) have found that when transactive reasoning/discourse was promoted in whole class discussions, there were positive implications for the students’ learning of proof. “Transactive reasoning is characterised by clarification, elaboration, justification, and critique of one’s own or [anothers’] reasoning.” (Goos, et al. 2002) In the study by Blanton and Stylianou (2014), teachers participated and focused the classroom discourse with various types of utterances. Transactive teacher utterances included requests for critiques, explanations, justifications, clarifications, elaborations, or strategies. They were in the form of questions that asked students for immediate responses requiring transactive reasoning. Other types of teacher utterances were facilitative (often rephrasing a student utterance), didactive (lecture), and directive.

Giving prompts on written work is asynchronous, as opposed engaging in verbal classroom discourse which occurs in real time (synchronously). Yet the goal is much the same, to get students not only to explain and justify where necessary, but to also reflect on their work and refine their proofs. In an inquiry oriented classroom, the asynchronous prompts and synchronous discussions are all part of the sociocultural approach to teaching (Goos, 1999; Goos, et al, 2002; Vygotsky, 1978). Looking at the type of prompts given in the pilot study, and the responses to those prompts lead us to believe that transactive prompts in the form of questions might be the most effective prompts in this context.

Framework

As far as we know, there is not an established framework within which to work, so we will be drawing on the work of Blanton and Stylianou (2014) in their verbal discourse analysis to start to categorize the types of written teacher prompts in the pilot data. We will initially look at the written prompts through a transactive/facilitative/didactive/directive lens. We will also look at the student pre and post work to identify whether the student appears to need to clarify or is on the wrong track entirely, drawing on Vygotsky’s notion of zone of proximal development (ZPD). Finally we will see if there appears to be any effect of the type of prompt given. We want to identify the most effective type of written prompt.

Our research question is: Do different types of written prompts affect students' constructions of proofs?

Methods and Very Preliminary Data:

Data collected in the pilot study by the Morrow during the spring 2015 semester will be coded and analyzed by both authors. Approximately 130 rough (and not so rough) drafts of proofs with teacher prompts and their associated final proofs were collected from that spring 2015 class. Additionally, during the fall 2015 semester, the Morrow will collect some additional written proof sketches (with prompts) and final proofs from her introduction to proofs class. Also, during the fall, 2015 semester, Shepherd will collect copies of written proof sketches (with prompts) and final proofs from her Abstract Algebra—Groups class. The initial coding scheme will be based on the Blanton-Stylianou teacher utterances classifications (transactive, facilitative, didactive, directive) for the teacher prompts during whole class discussion. Additionally, a coding for the assumed need the student should address from the pilot study will be jointly developed guided by Vygotsky's ZPD ideas. Is the student on the "right track?" What is needed by the student to progress? Is the student is on a "wrong track" what does the student need to move toward a more productive line of thinking. This two track coding scheme, in addition to being used to analyze the data collected last spring, will be applied to the new data received this fall. Each teacher will initially code her own students, then the other teacher's students. Differences in coding will be discussed and adjustments made so that a consistency in coding can be developed. In addition, there will be an assessment of the students' final proof attempts to decide the effectiveness of the initial teacher prompt.

It is expected that the coding will have to be revised throughout the coding process as possible unanticipated patterns in prompts or responses occur. Very preliminary analysis of 5 examples from Morrow's spring 2015 Abstract Algebra class seems to indicate that transactive questions (utterances that are both transactive and posed as a question) are more effective. Two examples are given below.

In example 1 we see a case where a transactive question prompt was posed and the final proof was essentially correct, along the same idea as the initial sketch (as opposed to being very different and essentially what some other student presented in class), and seems to show the teacher prompt was effective.[1] This teacher prompt is considered a transactive question since it asks the student to clarify a statement.

Example 1: **Problem 93**. *Suppose that G and H are groups, and that $\phi: G \to H$ is an isomorphism. Prove that if G is abelian, then H is abelian.*

(initial student proof with prompts)
Let $\phi: G \to H$ be an isomorphism.
Let G be abelian.
Then, for all $a, b \in G$, $ab = ba$.

Let $c, d \in H$ such that $\phi(a) = c$
and $\phi(b) = d$.

[1] The teacher prompts are the drawn and italicized parts.

Then, $cd = \phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a) = dc$.

Therefore, since $cd = dc$, H is abelian.

? Clarify ?
Where does this a come from.
Can you convince me that such an a exists?

(final student proof)
Let $\phi: G \to H$ be an isomorphism.
Let G be abelian.
Then, for all $a, b \in G$, $ab = ba$.

Let $c, d \in H$.
Since ϕ is an isomorphism, it is onto
So there exist $a, b \in G$ such that
$\phi(a) = c$ and $\phi(b) = d$.

Then, $cd = \phi(a)\phi(b) = \phi(ab)$.
Since G is abelian, $ab = ba$ so
$\phi(ab) = \phi(ba) = \phi(b)\phi(a) = dc$.

Thus, since $cd = dc$, H is abelian.

Example 2 shows a teacher prompt where the revisions are not as good as they should have been. Some of the errors are fixed. The first set of prompts, both involving the incorrect operation in the group G, would be considered a transactive question, and effective. The longer (italicized) prompt is more directive, telling the student what needs to be done and is not effective. It would appear the student did not re-engage with the material in a transactive fashion to address showing that any element of the group is a power of a^k.

Example 2: *Suppose $G = \langle a \rangle$, where a has finite stack-height (order) m. Prove that if k is an integer relative prime to m, then a^k is also a generator of G.*

(Initial student proof with prompts)
Let k be an integer relative prime
to m. Then there exists integers
s and t such that

$$sk + mt = 1$$

? operation in G?

Then,

$$a^{sk} + a^{mt} = a^1$$

(?)

So, $a^{sk} + e = a$

$$a^{sk} = a$$

So $a^{ks} = a$ *aha*
$(a^k)^s = a$

Then, a^k is a generator
for G, since $a^k$to a power
is equal to the generator a.

Yes – that's the key idea…
but you you still need to turn it
into mathematical proof –
the main part is to show that
<u>IF</u> $x \in G$, then $x = (a^k)^n$for some
integer n.

(final student proof)
Let $a \in G$.
Then $a^{sk+mt} = a$, note $a = a^1$.

So, $a^{sk} \cdot a^{mt} = a$
Then $a^{sk} \cdot (a^m)^t = a$
So, $a^{sk} \cdot (e)^t = a$
And $a^{sk} \cdot e = a$
So, $a^{sk} = a^1$, note that $sk = ks$.

Then $a^{ks} = a$

So $a = (a^k)^s$

Then, $a^k$is a generator for G
since a^k raised to a power is
equal to the generator a.

Preliminary results will be presented at the 2016 RUME Conference, and further questions and research will be designed for the spring 2016 semester so that a more complete framework can be constructed. It is anticipated that the data will show that teacher prompts in the form of transactive questions are more effective in helping students construct proofs.

Questions for discussion

1. Is the transactive, facilitative, didactive, directive scheme used the most appropriate for this type of analysis?
2. If indeed transactive questions are more effective, how can we train teachers and TAs in their use?

References

Blanton, M. L. and Stylianou, D. A. (2014). Understanding the role of transactive reasoning in classroom discourse as students learn to construct proofs. *Journal of Mathematical Behavior, 34,* 76-96.

Goos, M. (1999). Scaffolds for learning: A sociocultural approach to reforming mathematics teaching and teacher education. *Mathematics Teacher Education and Development, 1,* 4-21.

Goos, M., Galbraith, P., and Renshaw, P. (2002). Socially mediated metacognition: Creating collaborative zones of proximal development in small group problem solving. *Educational Studies in Mathematics*, *49,* 193-223.

Harel, G. and Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In Schoenfeld, A., Kaput, J. and Dubinsky, E. (Eds), *Research in Collegiate Mathematics Education III* (pp. 234-283). Providence: AMS.

Johnson, E. (2013). Teachers' mathematical activity in inquiry-oriented instruction. *Journal of Mathematical Behavior, 32,* 761-775.

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics,* 52, 319-325.

Remillard, K. S. (2014). Identifying discursive entry points in paired-novice discourse as a first step in penetrating the paradox of learning mathematical proof. *Journal of Mathematical Behavior, 34*, 99-113.

Sriraman, B. (2004). The characteristics of mathematical creativity. *The Mathematics Educator, 14,* 19-34.

Speer, N. M., Smith, J. P., Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *Journal of Mathematical Behavior, 29*, 99-114.

Vygotsky, L. S. (1978). *Mind in Society*. Harvard University Press, Cambridge, MA.

Weber, K. (2001). Student difficulty in constructing proofs: the need for strategic knowledge. *Educational Studies in Mathematics*, 48, 101-119.Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25–41). Dordecht: Kluwer.