Students' meanings of a (potentially) powerful tool for generalizing in combinatorics

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In this paper we provide two contrasting cases of student work on a series of combinatorial tasks that were designed to facilitate generalizing activity. These contrasting cases offer two different meanings (Thompson, 2013) that students had about what might externally appear to be the same tool – a general outcome structure that both students spontaneously developed. By examining the students' meanings, we see what made the tool more powerful for one student than the other and what aspects of his combinatorial reasoning and his ability to generalize prior work were efficacious. We conclude with implications and directions for further research.

Key Words: Combinatorics, Generalization, Counting problems, Mathematical meanings

Introduction and Motivation

Generalization is a fundamental mathematical activity with which students at all levels engage (Amit & Klass-Tsirulnikov, 2005; Lannin, 2005; Peirce, 1902), and yet there is still much to learn about ways to foster productive generalizing activity. In particular, most of the work on generalization has been with younger children, commonly in algebraic settings (Amit & Neria, 2008; Becker & Rivera, 2006; Cooper & Warren, 2008; Ellis, 2007b; Mulligan & Mitchelmore, 2009; Radford, 2006; Rivera, 2010; Steele, 2008). In the context of a larger study, we sought to better understand students' generalization in the domain of combinatorics which involves the solving of counting problems and provides students with opportunities to engage with accessible yet challenging tasks (e.g., Kapur, 1970; Tucker, 2002). In this paper, we compare and contrast two students' work on a series of combinatorial tasks, during which they each spontaneously introduced a potentially powerful tool for generalization in the combinatorial setting. Each of these students used this new tool, but they varied in the meaning they seemed to make of the tool. As a result, they differed in how effective they were able to be in using the tool generally and solving combinatorial tasks. We seek to answer the following research question: *What meaning do students make of the same spontaneously generated tool (which we refer to as the 11xx structure), and what do these meanings suggest about students' generalization in combinatorial contexts?* The results in this paper help to inform research on students' meanings in the context of both their generalizing activity and their combinatorial thinking.

Literature Review and Theoretical Perspective

The act of generalizing is a key aspect of students' mathematical development, and both mathematics education researchers (e.g., Amit & Klass-Tsirulnikov, 2005; Davydov, 1972/1990; Ellis, 2007b; Vygotsky, 1986) and policy makers emphasize its importance (the Common Core State Standards highlight generalization in both the content and the practice standards; Council of Chief state School Officers, 2010). We seek to extend the current work on generalization by focusing on undergraduate students in the context of combinatorics. The tasks we designed were designed with the overall aim of facilitating students' generalizing activity, and for this purpose we follow Ellis (2007a) (who drew on Kaput, 1999) in defining generalization as "engaging in at least one of three activities: a) identifying commonality across cases, b) extending one's reasoning beyond the range in which it originated, or c) deriving broader results about new relationships from particular cases" (p. 444). We chose the context of combinatorics in part to examine generalization in a novel context, but we were also motivated to contribute to previous work on students' combinatorial thinking. There is evidence that students struggle with solving counting problems correctly (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Hadar & Hadass, 1981), and we hope to contribute to the existing literature by providing instances of meaningful combinatorial reasoning that might ultimately inform instruction.

We draw on Thompson's (2013) notions of meanings in this paper. He argues for the importance of developing meaning of the idea meaning (p. 57) and that a greater emphasis on mathematical meaning could contribute to a more coherent educational experience for students overall. Thompson surveys different meanings of "meaning," and we adopt his alignment with a Piagetian view of meaning as assimilating a scheme (p. 60). Thompson notes that, "From a Piagetian viewpoint, to construct a meaning is to construct an understanding – a scheme, and to construct a scheme requires applying the same operation of thought repeatedly to understand situations being made meaningful by that scheme" (p. 61). Also, importantly, Thompson emphasizes meaning from the students' perspective:

> "The meanings that matter at the moment of interacting with the students are the meaning that students have, for it is their current meanings that constitute the framework within they operate and it is their personal meanings that we hope students will transform" (p. 62).

For this reason, in this paper we seek to understand and interpret students' meanings in order to gain insight about what made their work particularly productive (or unproductive) in the contrasting cases. We use this notion of meanings in this paper because we have a situation in which two students introduce and use a tool that externally seems very similar, but their different meanings of that tool cause them to use it differently. We thus find it useful to discuss the variety of meanings students had about what appears to be a very similar mathematical phenomenon.

Methods

In this study we conducted a set of single, individual, hour-long interviews with ten calculus students as they worked through what we call the Passwords tasks, and in this paper we report on two contrasting cases of students' work. We chose students who had not been taught combinatorics formally at the college level. The main goal of the tasks was to put students in a situation in which we could evaluate their generalizing activity as well as gain insight into their combinatorial reasoning. The progression of Passwords tasks is as follows:

First, we had students solve the problem, *How many 3-character passwords can be made using the letters A and B?*, and we explicitly directed them to organize their work by completing tables according to the number of As in the password. We had them begin with 3-character passwords, and then also make tables for 4-character then 5-character passwords (Figure 1 shows Tyler's table for the 4-character AB passwords). There were some opportunities to observe generalization in building up these cases, as students could observe relationships and similarities among the tables or could make combinatorial observations that held across cases.[1] We wanted the students to build (typically through partial or complete listing) the tables to see how they would use them as we progressed to the next part of the tasks.

[1] The reader may note that the emerging tables involve binomial coefficients, or the ways of selecting positions for the A's to go, as the remainder of the positions must consist of B's (these entries also correspond to rows in Pascal's triangle). We did not expect students to recognize binomial coefficients or even to conceive of these entries as involving choosing in any way, and indeed most students did not. Eventually, we could increase the *number* of numbers in the passwords, and ultimately move toward developing a formal statement of the binomial theorem (which we accomplished with one student, not discussed here).

Then, we moved onto passwords involving the number 1, and the letters A and B. We had students make tables for 3-character and 4-character passwords, organized according to the number of 1s in the passwords (Figure 6 shows Tyler's table for the 4-character 1AB passwords). Note that we can use the previous tables in the following way: we can think first of determining positions for the 1s (which the previous AB table provides), and then the problem is reduced to counting passwords involving only A's and B's. For example, in making a table for the number of 4-character passwords with 1, A, and B, for each respective row of the table (0, 1, 2, 3, or 4 1's), we can first think of counting the number of ways of placing the ones. There are 1, 4, 6, 4, 1 respective ways of doing this, which is reflected in the previous 4-character AB table. Once this is established, note that for each row in the table, once the ones are placed it is just a matter of counting passwords of length 3 using only A's and B's, reducing the problem to a previous problem (specifically, there are 2^3 such passwords). The point is that it is possible, with some combinatorial insight and understanding of the outcomes' structure, to leverage the previous work from the AB passwords in the more complicated 1AB passwords case. The interviews were videotaped and transcribed, and overall the videos and transcripts were analyzed so as to construct a narrative about the teaching experiment (Auerbach & Silverstein, 2003). We discussed the two contrasting cases with the entire research team and together formulated hypotheses about the students' meaning in each case via repeated viewings of video and reading of enhanced transcripts.

Results

In presenting our results, we describe different students' meanings of the same phenomenon. We highlight these results both to show an interesting phenomenon that emphasizes a potentially powerful tool toward meaningful combinatorial generalization and to suggest that students might need to ascribe certain meanings to such tools in order to leverage them in an impactful way.

Example 1 – Tyler. We begin with Tyler, who demonstrated an ability to reason comfortably and easily with outcomes. His method of solving the tasks typically involved some organized listing. For example, in trying to determine the number of 4-character AB passwords with exactly two A's, Tyler made the list in Figure 2 and gave the following explanation:

Tyler: *Um. Yeah I guess I started with the first one being A um, and then I did like 2 A's consecutively and then B's, and then moved the B over one, and then, um, moved the next B over one… And then, after that I just start with the B and kind of did the same thing.*

He ultimately correctly created the table for 3, 4 (Figure 1), and 5-character AB passwords.

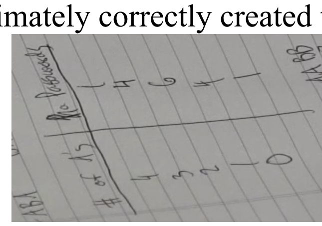

Figure 1

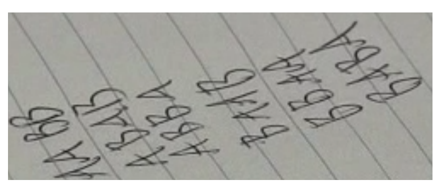

Figure 2

Early in the interview, Tyler had established that there were a total of 2^n n-length passwords using only A's and B's. He established this primarily through noticing a numerical correlation after giving the totals for 3, 4, and 5-length passwords, read from his empirical tables (noticing the 3-character AB passwords table had 1+3+3+1 = 8 total passwords). He went on to write the relationship "*n length = 2^n combos*," but we suspect that he did not meaningfully understand the multiplication principle as a combinatorial way of explaining the expression 2^n.

We then moved on to counting passwords that consist of characters 1, A, or B. Tyler felt that there was more to keep track of, but he persisted with listing outcomes and filling out the table as he had in the previous situations. He managed to list the entire table for the 3-character 1AB passwords, and again he used systematic listing used to do so, and he seemed to maintain a combinatorial understanding of the entries in his tables.

Next, we asked him with to fill out a table with the four-character 1AB passwords (organized according to number of 1s). He got started but paused and said, “I can’t really think of any pattern,” and he seemed to realize that this table would be more difficult to work out than the previous case. We directed him to perhaps start thinking about the rows for zero and one 1’s (starting from that end of the table). Tyler then did something unexpected – he introduced a way of describing a general outcome involving 1’s and x’s, as seen in the following exchange.

Tyler: *Yeah. Ok so the 0 [row] was gonna be, what did I come up with here [refers back to the 4-character AB table] 10, 15, 16 if uses, um. And then the 1, so what I was thinking, what I was saying earlier. How there is only a certain amount of spots for it, like it has to be, like I’m just gonna use x cause, um, has to be in one of these spots... [draws Figure 3]*

Int.: *Great.*

Tyler: *So there’s, now there’s just 3 x’s um, and I know that for…3 spots with 2 different letters there’s going to be 8 different ways to do it [points back to the previous 3-character AB table, see Figure 4]…Um so I guess 8…there’s 8 different of each of those just using this same table umm, there’s just 32 so I want to say there’s gonna be, um, 32 for just the 1.*

Int.: *Okay and you got, you’re thinking of that as kind of the 4 times 8?*

Tyler: *Yeah I, just adding them all up.*

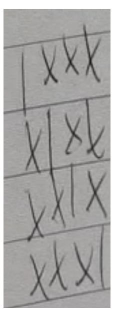

Figure 3

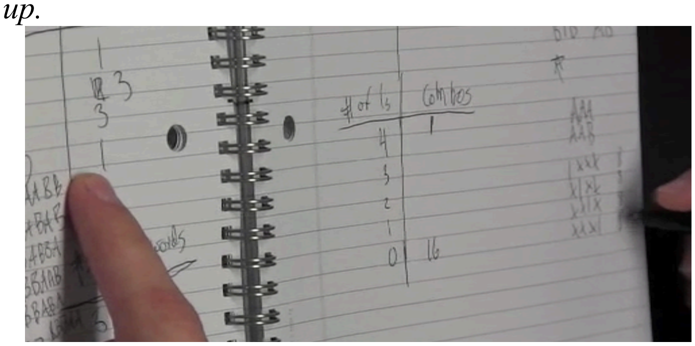

Figure 4

This was a key moment in Tyler’s work. He spontaneously introduced a very powerful tool for how to count desirable passwords in the form of a general structure consisting of x’s and 1’s. (For ease of communication, we hereafter refer to the tool as “the *11xx* structure,” which is meant to suggest the introduction of the variable of x as a means of representing a more general outcome.) We contend that this was a general representation of an outcome (a password), perhaps a product of his rich facility with listing. He realized that in each case where there was a 1 with three x’s, there would be 8 such possibilities (because there were 8 total 3-character AB passwords), and his total would be 32. He was thus able to recognize that he could use his previous case as a part of the more complicated new situation. We can further explore this moment of insight as he continued to use the *11xx* structure in filling out the rest of the 40-character 1AB table. Figure 5 shows his listing of x’s and 1’s in the four-character 1, A, B case, with exactly two 1’s. There are exactly 6 of them, and the following exchange demonstrates Tyler’s meaning of those six general outcomes as they relate back to his previous work. Specifically, note that he understands why 6 such outcomes would make sense, because he can understand that he is in a situation of arranging two distinct objects, which is what his previous work involving AB passwords also entailed.

Tyler: *Yeah there you go. Is that all of them? Yeah so 6, ‘cause that would make sense...*

Int.: *Does that 6 make sense?*

Tyler: *Does it? Uh, well that would that's um, 2 variables like instead of doing 3 things there's 2 umm, with the 4 combo, so 2, was 6 over here [points back to the 6 in the correct entry of the 4 character AB password table], so that's why I thought it made sense.*

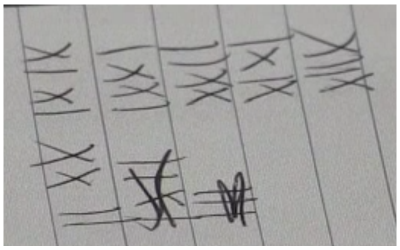

Figure 5 – Tyler's

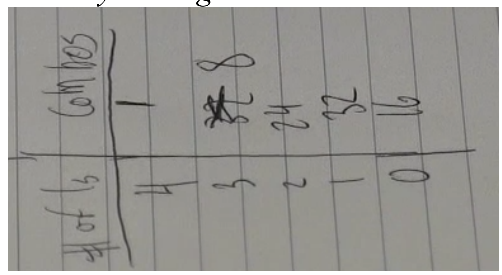

Figure 6

He continued to work in a similar fashion for the case of three 1's, and he ultimately arrived at the correct table for 4-character 1AB passwords (Figure 6). Although we do not have space to outline his subsequent work, Tyler did go on to use the same tool in subsequent cases involving 5-character 1AB passwords. He seemed to have a robust understanding of how he could use the tool to solve password problems involving more characters and more letters.

We point out a couple of important features of Tyler's work. First, it is noteworthy that Tyler spontaneously introduced a new, general structure that appropriately represented the situation and the outcomes he was trying to count. This is in and of itself impressive, and his work demonstrates an existence proof of the kind of thinking the Password tasks fostered in terms of combinatorial generalization. Second, Tyler was able to relate that new structure to his combinatorial activity to that point, and this relationship to prior work played a key role in him ultimately being able to solve more problems correctly. Importantly, he seemed to preserve the combinatorial meaning of the tasks and the situations as he related the *11xx* structure with his previous work. In terms of Tyler's meanings, we interpret that he understood the *11xx* structure as a general structure of the outcomes he had been working with, allowing him to relate back to a previous combinatorial situation involving just two objects (specifically, A's and B's). Although he did not demonstrate a deep combinatorial, multiplicative meaning of 2^n, he could recognize the 2^n as being numerically equivalent to a previous case, which he could use effectively.

Example 2 – Richie. We now contrast Tyler's work with another student, Richie. Richie, too, spontaneously introduced the *11xx* structure, but we highlight a key difference in that Richie was less successful in leveraging the new tool by relating it to previous circumstances. When making the tables for the AB passwords tasks, Richie correctly filled out the tables, often using some listing, but it seemed as though he was more attuned to the numerical patterns he observed than in finding a combinatorial explanation that made sense. For example, when making a table for the 5-characer AB passwords, we had the following exchange. Notice that his justification for why certain entries were in the table was based on the patterns he'd observed. This is not in and of itself problematic, but it shows perhaps that he was not establishing a robust combinatorial meaning but that his meaning was based on observed numerical regularity.

Richie: *So when you get to like the -- the second one, or it's not even like the second one, it's more like the one in the middle of 0 and 5 is going to be the most possibilities. And in previous problems it's been like 2 more than the preceding one.*

Int.: *Okay. Sure.*

Richie: *And I'm just assuming that this is 5, because the previous pattern's like increasing by 1.*

When we moved on to the 1AB passwords case, Richie, like Tyler, spontaneously moved toward a new structure involving 1's and x's. Figure 7 shows his drawing of the six ways to arrange two 1's and two x's.

Richie: *I'm just trying to think of all the different configurations – that 1s can (inaudible) – so like they can be starting with X, or just like that. Or like this or like this. Or like this. And then it could be 1 inside.*

Int.: *Perfect.*

Richie: *So, 1, 2, 3, 4, 5, 6. There's 6 different possibilities for that. And then each of these can have 4 different configurations.*

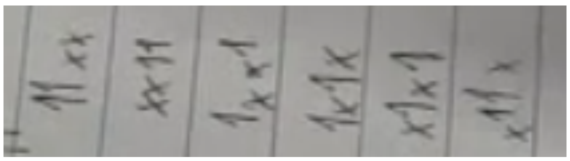

Figure 7

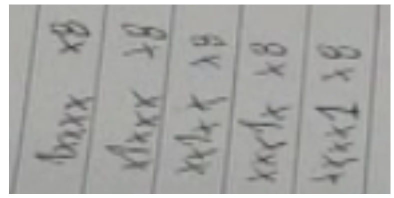

Figure 8

Richie then checked his work and reasoned that for each of those possibilities there would be four possibilities, running through BB, AA, AB, and BA. He concluded, "So 6 times 4 would be – it would be like 6 times all of these really [referring to the six configurations]…Yeah, okay. So I guess they would be 24. So it would be 24 possibilities." Richie then continued to work, and as he progressed to other rows in the table he made more diagrams involving 1's and x's.

We then had him move to the 5-character 1, A, B password, and I asked him to start making the table. Here again he made a similar diagram with 1's and x's, but here his work departed from Tyler's. Richie was able to think about there being a certain number of options for each case (each arrangement of the x's), and he knew there were two options for each x (A and B), but he added instead of multiplied the number of options, yielding 8 rather than 16 possibilities.

Richie: *So for 5 it would be 32. Same thing. And then for 1 there would be – (writes Figure 8 without the *8's) and so those would each have – this could be A or B, so that would be 2 for that, 2 for that, 2 for that, so these would each have 8 different possibilities [writes the *8's in Figure 8]. So it would be 5 – 5 times – it would be 40 for 1.*

In asking Richie to explain this work, we gain insight into his meaning of the diagram. He made no explicit connection to the previous tables or situations as Tyler had.

Richie: *This, like I made I want to say like a diagram basically of a position so 1 can be. And then I put Xs in for the – where the As and Bs could be, because those are variables that can be either A or B. And then I noticed that for every X it has 2 possibilities, either being A or B, and there's 4 Xs…So then I just multiply that by 2 to get 8. So each – for each 1 position there's 8 different possibilities for the password. And that's how I got 40.*

Richie continued his work and listed out all 10 of the configurations of two ones in a 5-character password. He demonstrated a consistent meaning by again adding the options – saying there were 6 passwords for each configuration, which is 2*3 rather than $2^3 = 8$.

At first blush it seems that perhaps Richie simply made a mistake, adding instead of multiplying the options, but we do not feel that he simply made a numerical error. Instead, the evidence seems to point to the fact that he did not make meaning of the new structure as being related to the previous case, at least not directly to the previous tables. Unlike Tyler, he did not recognize that the power of the *11xx* structure is that it can be very clearly related to the previous situation. There are two potential points of connection to the prior AB tables (relating the placement of the 1's to the rows of the AB tables, such as 1, 4, 6, 4, 1, and realizing the totals in the previous tables represent the possible number of passwords of a given length), both of which Tyler recognized, and neither of which Richie recognized.

This is not to say that Richie's meanings were unreasonable or that they did not make sense to him – indeed they did. His introduction of the *11xx* structure seemed to serve as a way of simplifying and organizing the problem so he could better break it down, but not in a way that facilitated rich connections to the previous problem. We asked him how and why he came up with the structure, and he suggested that he was motivated by efficiency: "I started to write here different configurations for where the 1s could be and where the Bs and As could be, and I noticed that basically the As and Bs were just switching places for wherever the amount of 1s were. So I started putting Xs there just so I wouldn't have to write as much."

Conclusion and Implications

By examining two students' meanings of the same tool that they each spontaneously developed, we gain insight both into students' generalizing activity and their combinatorial reasoning. Ultimately, we want to help students be more effective in creating productive generalizations, and we want to learn more about how students might effectively solve counting problems. We feel that Tyler's work – not only his production of the *11xx* structure but his ability to make meaning of it in light of prior activity – is a powerful example of a student-generated general structure that led to inroads in challenging combinatorial tasks. Set in contrast to Richie's work (which was also impressive in that he generated the *11xx* structure, but was limited in its lack of combinatorial meaning and connection to the previous situations), we can examine what aspects of Tyler's work and meanings were so efficacious. One aspect of his work that was powerful was that he remained grounded in his prior activity, and he had a rich combinatorial meaning of those prior situations. The AB tables Tyler made were combinatorially meaningful for him, in the sense that he reasoned about outcomes and did not lose sight of the combinatorial context. This is in line with previous work that emphasizes the importance of outcomes (e.g., Lockwood, 2013; 2014). We suspect that because Tyler had such a strong sense of outcomes (as seen through his listing activity in his creation of AB tables), the *xx11* structure really did represent to him a more general form of an outcome. It resembled a password (still a sequence of characters on the page), and we posit that this enabled him to maintain his reasoning about the structure of his outcomes and thus a connection to the previous combinatorial situation.

In terms of implications, our findings suggest that students can, on their own, produce potentially powerful tools involving general structures. However, this alone is not sufficient for productive generalizing or counting activity, and these contrasting cases show some of the other reasoning necessary to make full use of such tools. A pedagogical implication is that teachers may need to be vigilant in helping students maintain contact their with their prior activity. Specifically, in combinatorics this might mean that even as students notice patterns, teachers should help them to connect those patterns to the combinatorial context and not simply to numerical regularity. Combinatorially, another implication of the work is that this sequence of tasks does seem to be potentially useful in helping students to reason about the binomial theorem (or at least its initial stages). Tasks like these could be leveraged to introduce and teach combinatorial identities, which is a building block toward the learning of combinatorial proof.

Our findings show an example of rich generalizing activity in a combinatorial context. These findings emerged in a single interview, but we hope to extend this work through teaching experiments in which students' meanings can be developed and examined over time. Next research steps also include an investigation into more specific instructional interventions that might foster the kind of meanings that proved beneficial for students like Tyler.

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