# Use of strategic knowledge in a transition-to-proof course: Differences between an undergraduate and graduate student

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*The ability to construct proofs has become one of, if not the, paramount cognitive goal of every mathematical science major. However, students continue to struggle with proof construction and, particularly, with proof by contradiction construction. This paper is situated in a larger research project on the development of an individual's understanding of proof by contradiction in a transition-to-proof course. The purpose of this paper is to compare proof construction between two students, one graduate and one undergraduate, in the same transition-to-proof course. The analysis utilizes Keith Weber's framework for Strategic Knowledge and shows that while both students readily used symbolic manipulation to prove statements, the graduate student utilized internal and flexible procedures to begin proofs as opposed to the external and rigid procedures utilized by the undergraduate.*

*Key words*: Mathematics Education; Strategic Knowledge; Proof by Contradiction

## Introduction and Overview

The ability to construct proofs has become one of, if not the, paramount cognitive goal of every mathematical science major (Schumacher & Siegel, 2015). However, students at all levels struggle with proof construction (Stylianou, Blanton, & Rotou, 2014), and in particular struggle with constructing proofs by contradiction (Brown, 2013). The purpose of this paper is to report on the results of a pilot study on student's understanding of proof by contradiction in a transition-to-proof course. In particular, this paper will address the following research question: Is there a difference in proof by contradiction strategies between two students, an undergraduate and a graduate student, enrolled in the same transition-to-proof course? The Strategic Knowledge framework, outlined in Weber (2004), will be used to analyze the strategies these students utilized in constructing proofs. The following section will give a brief overview of the Strategic Knowledge framework.

## Strategic Knowledge Framework

Weber (2004) developed a framework for describing undergraduate proof construction processes based on the observations of 176 undergraduate students' proofs over multiple studies. This framework classified the types of proofs produced as one of the following: procedural, syntactic, or semantic.

In a proof using a procedural method, "one attempts to construct a proof by applying a procedure, i.e., a prescribed set of specific steps, that he or she believes will yield a valid proof" (Weber, 2004). The procedure can either be an algorithm or a process. Algorithms are characterized as external and highly mechanical to the student, whereas a process is internal and flexible. By external, it is meant the procedure came from outside of the student, such as from an instructor. By internal, it is meant the procedure has been interpreted and constructed by the individual.

In a proof using syntactic methods, "one attempts to write a proof by manipulating correctly stated definitions and other relevant facts in a logically permissible way" (Weber, 2004). Proofs of this form are no more than unpacking definitions and using tautologies to manipulate symbols to achieve the desired conclusion. Students using this method do not need to consider the meaning of their syntactic statements.

In a proof using semantic methods, "one first attempts to understand why a statement is true by examining representations (e.g., diagrams) of relevant mathematical objects and then uses this intuitive argument as a basis for constructing a formal proof" (Weber, 2004). Very few undergraduate research subjects, if any, attempted semantic proofs; 0 of 56 proofs in abstract algebra and 17 of 120 proofs in real analysis.

## Methodology

This case study is situated in a larger research project on the development of an individual's understanding of proof by contradiction in a transition-to-proof course. *Bridge to Higher Mathematics/Thinking Mathematically: Intro to Proof* is the first course in which students are formally introduced to mathematical proofs and their accompanying methods at a large, public university in the southeastern United States. Data for this report consists of written student attempts to prove three number theory statements[1] as well as individual interviews detailing their thought process while constructing the proofs.

Two students volunteered to be interviewed in Spring 2015: one undergraduate and one graduate student. The undergraduate, James, is a double major, in Computer Science and Mathematics, while the graduate, Frank, is an Economics major. Despite the difference in degree program, both James and Frank have completed similar mathematics courses and can be considered to have similar mathematical backgrounds.

## Data Analysis

A problem-by-problem analysis of the two interviewees using the Strategic Knowledge framework follows. This analysis will begin with an overview of their exhibited proof strategies for the problem, followed by a copy of their written proof for the problem, and ending with an in-depth analysis utilizing the participants' responses during the interview. Due to page limitations, analysis of only two of the three proofs will be provided.

To code proof methods, the following guidelines were used. First, any mention or consideration of the meaning of a mathematical statement was coded as "semantic". If there was no mention or consideration of the meaning of the mathematical statements and the proof was primarily written with symbolic manipulation, the proof was coded as "syntactic". For the remaining methods, rigid (i.e. specific to the particular problem) and external (i.e. rules set by another party) methods were coded as "algorithmic" and flexible (i.e. adaptable to a range of problems) and internal (i.e. synthesized rules for the individual) methods were coded as "process".

**Problem 1: If *a* is an irrational number, then *a+2* is an irrational number.**

For problem one, James began with an algorithmic approach to the proof. Once he converted the statement to symbolic notation, he then primarily used a syntactic approach to complete the proof. At no time during the proof did he exhibit or profess a semantic approach to the statement. James' written work for problem one is displayed in Figure 1 below.

[1] All three statements could be proved by contradiction, though contradiction was not necessary.

1. If a is an irrational number, then a+2 is an irrational number.

Negation: If $a+2$ is a rational #, then $a$ is a rational #.

Proof (by contrapositive): Suppose $a$ is a PAC ~~integer~~ # (particular but arbitrarily)

WNTS: $a$ is a rational #.

Let $a+2 = \frac{b}{c}$, where $b, c$ are some integers s.t. $c \neq 0$. (by def. of rational #s).

Then $a = \frac{b}{c} - 2$ (by substitution).

$= \frac{b-2c}{c}$ (by finding a LCD of $c$ & multiplying 2 by $c$)

Since $b-2c$ is a difference of integers, then it is an integer; also, $c \neq 0$

Then by def. of rational #s, $a = \frac{b-2c}{c}$ is rational.

Since the negation is true, then $\forall a \in \mathbb{R}$, $a$ is an irrational # $\Rightarrow$ $a+2$ is an irrational #.

Figure 1: James' Proof for 1st Statement

When asked how he started the proof, James stated "So I guess I did more practice on them [proofs by contrapositive], during Discrete and Bridge, that's where I got used to it." James' use of the phrase "I got used to it" indicates a passive and external role in writing proofs by contrapositive. When asked why he chose contrapositive, he continued to repeat that he uses contrapositive with "these types of proof"; his inability to articulate exactly what this type of proof was illustrates the external nature of why he completed the proof as he did.

For problem one, Frank utilized a syntactic method to write the proof by converting the statement to symbolic notation, after which he manipulated the symbols to complete the proof. He also displayed a flexible procedure for proof by contradiction, though at no time during the proof did he exhibit or profess a semantic approach to the statement. Frank's written work for problem one is displayed in Figure 2 below.

Prove the following statements using any proof techniques.

1. If a is an irrational number, then a+2 is an irrational number.

$a$ is not $\mathbb{Q}$ $\Rightarrow$ $a+2$ is not $\mathbb{Q}$

$a$ is not $\mathbb{Q}$ $\wedge$ $a+2$ is $\mathbb{Q}$

Pf (By CONTRADICTION):

Suppose $a+2$ is rational.

Then $a+2 = \frac{p}{q}$ for integers $p$ & $q$, $q \neq 0$. (DEF OF RATIONAL #s)

$\Rightarrow a = \frac{p}{q} - 2$ (LAWS OF ALGEBRA)

$\Rightarrow a$ is a rational number as $\frac{p-2q}{q}$ is rational as $p-2q$ is an integer by sums of integers being integers

→ But this is contradictory to the given that $a$ is not rational.

$\therefore$ the original is true. ☒

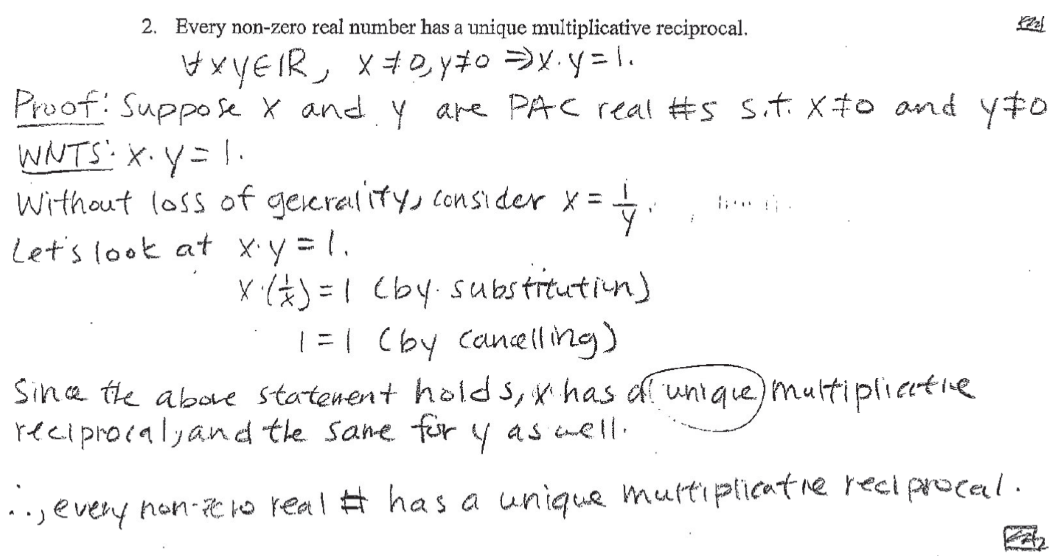
Figure 2: Frank's Proof for 1st Statement

When asked how he started the proof, Frank stated "I basically set it up so that I could say that *a+2* is rational and solved it out and said that by subtracting the two to the other side, you would still get a rational number and then you would get *a* is rational, which is not true because of the givens." This flexible overview of his proof is evidence of procedural knowledge and, in particular, a process for proving statements by contradiction.

As evidenced above, Frank began the proof by converting the statement to be proven into prepositional logic notation and mainly uses syntactic methods to continue in the proof. He does not consider the meaning of the statement, evidenced by his explanation: "But I think once I got here [Suppose *a+2* is rational], it was very obvious that I could just solve it out."

**Problem 2: Every non-zero real number has a unique multiplicative reciprocal.**

For problem two, James utilized a syntactic approach for the entire proof. However, he showed a procedural approach to the proof in general through his structure and reliance on definitions to fill the holes of the syntactic method. During the discussion of his proof, James showed he explicitly did not use a semantic approach to the statement. James' written work for problem two is displayed in Figure 3 below.


2. Every non-zero real number has a unique multiplicative reciprocal.

$\forall x y \in \mathbb{R},\ x \neq 0, y \neq 0 \Rightarrow x \cdot y = 1.$

Proof: Suppose $x$ and $y$ are PAC real #s s.t. $x \neq 0$ and $y \neq 0$.

WNTS: $x \cdot y = 1$.

Without loss of generality, consider $x = \frac{1}{y}$.

Let's look at $x \cdot y = 1$.

$x \cdot (\frac{1}{x}) = 1$ (by substitution)

$1 = 1$ (by cancelling)

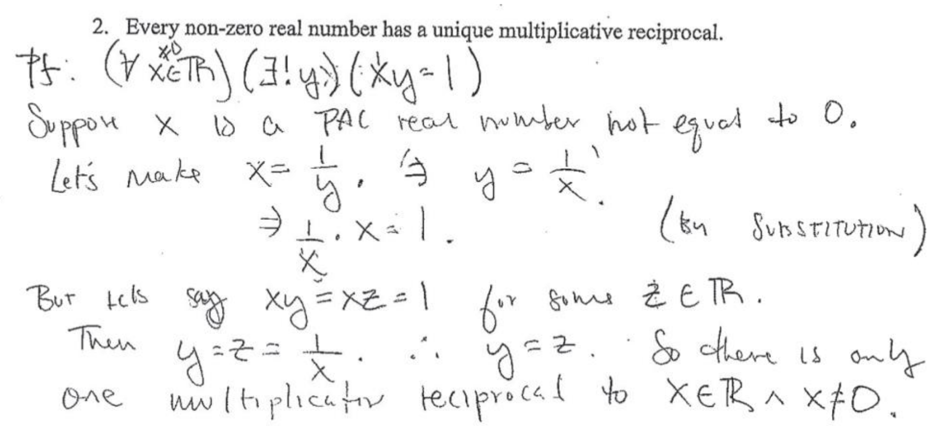
Since the above statement holds, $x$ has a unique multiplicative reciprocal, and the same for $y$ as well.

∴, every non-zero real # has a unique multiplicative reciprocal.

Figure 3: James' Proof for 2nd Statement

James' structure of proof highlights an external procedure to proving the statement. When James cannot prove a statement by symbolic manipulation, he relies on definitions. For example, in the proof above, James makes no justification as to why this reciprocal is unique. When probed whether he used the multiplicative inverse of *x* is *1/x* by definition, he says "Is that a definition? That's not a definition, is it? I don't think it is a definition, in my opinion." However, when probed specifically about why the reciprocal is unique, he states "Because *x* is unique, right? So it is a unique, a unique multiplicative inverse." As no other justification was conveyed, it must be by definition of a multiplicative inverse. This reliance on definitions can thus be seen as an external rule to justifying a statement when a justification is unknown.

For problem two, Frank utilized a syntactic approach for nearly the entire proof. However, he showed a procedural approach to the existence statement. At no time during the proof did he exhibit or profess a semantic approach to the statement. James' written work for problem two is displayed in Figure 4 below.


2. Every non-zero real number has a unique multiplicative reciprocal.

Pf: $(\forall x \in \mathbb{R}, x \neq 0)(\exists! y)(xy = 1)$

Suppose $x$ is a PAC real number not equal to 0.

Let's make $x = \frac{1}{y}$. $\ni$ $y = \frac{1}{x}$.

$\Rightarrow \frac{1}{x} \cdot x = 1$. (by SUBSTITUTION)

But lets say $xy = xz = 1$ for some $z \in \mathbb{R}$.

Then $y = z = \frac{1}{x}$. $\therefore$ $y = z$. So there is only one multiplicative reciprocal to $x \in \mathbb{R} \wedge x \neq 0$.

Figure 4: Frank's Proof for 2nd Statement

Frank began his proof by rewriting the statement in symbolic notation, just as he did in problem one. When explaining how he solved the proof, he stated "For number 2, I … basically put it into a more mathematical format. And then I … did some scratch work to solve for what the multiplicative reciprocal would be." Again, Frank relies on symbolic manipulation to proceed in the statement. However, when asked what type of proof this is, Frank said it was a direct proof. While it was suggested that multiple proofs could be combined, Frank used process of elimination to say the proof did not use contradiction, contrapositive, or induction. Since Frank still successfully proved that the multiplicative inverse is unique with a proof by contradiction (not explicitly), it can be said he has an external procedure to prove the existence of a mathematical object.

## Discussion

While both James and Frank used syntactic methods to prove this statement, James relied on rigid, external procedures to each problem to begin proofs, whereas Frank relied on flexible, internal approaches to begin proofs. When possible, both participants utilized symbolic manipulation, and thus exhibited a (productive) use of syntactic knowledge. Furthermore, since participant thought about the meaning of the mathematical concepts in the statements, we conclude that neither used semantic knowledge in their proof constructions.

This case study of two students, one undergraduate and one graduate, builds on the results of Weber (2001), in which Weber interviewed four undergraduate and four doctoral mathematics majors to examine differences in their proof construction. In this case study, the students have similar mathematical preparation and yet, the graduate student utilizes processes exhibited by the doctoral mathematics students in Weber's research. While it is reasonable to expect a difference in proof construction between students with different mathematical background, it is not clear why there should be a difference between students with the same mathematical background and different levels of program. Therefore, more research is needed to examine the differences between undergraduates and graduates with the same mathematical background with respect to their proof construction.

## Questions for the Audience

- How does major affect the types of strategic knowledge used to construct proofs?
- How does strategic knowledge fit within a student's proof schema?
- How much of an issue is the concept of infinity when students write contradiction proofs?

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