Fostering teacher change through increased noticing: Creating authentic opportunities for teachers to reflect on student thinking

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This paper reports results from a case study focusing on a secondary teacher's sense-making as she was challenged to reinterpret her meanings for algebraic symbols and processes. Building from these opportunities, she redesigned lessons to gather information about how her students conceptualized quantities and how they thought of variables, terms, and expressions as representing those quantities' values. She then used this information to respond productively to her understanding of individual students' meanings and reasoning elicited during these lessons. We argue that this case study demonstrates the potential for coordinating quantitative reasoning with teacher noticing as a lens to support teacher learning and we recommend specific mathematical practices that can help teachers develop more focused noticing of students' mathematical meanings during instruction.

Key words: Professional Development; Quantitative Reasoning; Teaching Mathematics

Introduction and Context

Many states adopted the new Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) as part of an effort to improve student learning and achievement. But research suggests that substantial change, at least in the short-run, will be difficult for the following reasons: (1) teaching practices and teachers' conceptions of their curricula are more significant indicators of student achievement than curricula alone (Boaler, 2003; Thompson, 1985), (2) teaching practices are learned implicitly as a cultural activity (Stigler & Hiebert, 2009), (3) teachers have a limited tolerance for the discomfort they feel while trying to implement reforms (Frykholm, 2004), and (4) reform efforts challenge teachers' traditional images of efficacy (Smith, 1996).

Smith (1996) and Thompson, Phillip, Thompson, and Boyd (1994) argue that thoughtful lesson planning that includes carefully conceptualizing learning goals, generating conjectures about how a person might come to understand certain ideas, and considering alternative solution methods may alleviate some of the uncertainty teachers feel in shifting their practice and might positively impact student learning. But our experience is that shifting teachers' practices is difficult. Interventions that only target a teacher's personal mathematics knowledge, such as providing additional upper level math coursework, might not translate to teachers modifying their established curricula and lessons. Some reasons for this may be that mathematics coursework does not advance teachers' knowledge of student learning of specific mathematics content; nor does it address challenges teachers encounter when determining and implementing curriculum for a particular course. Interventions with broad goals, such as shifting the emphasis from direct instruction to student-centered activity, often fail to impact student learning positively if teachers make only superficial changes in their teaching (Stigler & Hiebert, 2009).

Theoretical Perspective

We agree with Thompson's (1993, 2013) assertion that meaning is not carried by symbols or pictures on a page but rather exists in the mind of an individual. Accordingly, at any moment during an interaction between two individuals there is a meaning in the mind of the person communicating and a meaning in the mind of the interpreter. In the moment of teaching, students' current mathematical meanings are critically important because these are the meanings students either build upon or transform to advance their mathematical understandings. We align with von Glasersfeld (1995) who argues that one person cannot transmit knowledge to another. An individual must construct knowledge and meanings for himself through personal experiences. Thus, it is ineffective to *tell* teachers what students think (even if we knew), how to respond to students' questions, or how to leverage student thinking to provide more meaningful mathematical experiences. Instead, we must help teachers to become fluent in revealing the variety of student reasoning existing in their classrooms and to recognize the utility in being sensitive to student thinking when planning and delivering instruction. Mason (2002) argues that professional development is always a personal endeavor and that change occurs when a person increases the scope as well as the specificity of what he notices while engaged in professional practice and uses this intentional noticing to inform his actions at relevant moments. The more a teacher engages in disciplined noticing, the more likely it is that the teacher constructs new images of how a student may conceptualize and represent a problem or idea.

If we can foster increased noticing of student thinking in teaching and learning situations, we believe that teachers will be more likely to set goals related to building models of student thinking and to see their own practice as a source of learning and personal professional development (Cobb & Steffe, 1983; Lage-Ramirez, 2011; Stigler & Hiebert, 2009; Teuscher, Moore, & Carlson, 2015; Thompson & Thompson, 1996). We further believe that a focus on what Thompson (1994, 2013) calls *quantitative reasoning* is a fruitful avenue for fostering increased and disciplined noticing. By focusing attention on quantities, their relationship to one another, and how to represent the values of quantities using expressions or reasonable operations for calculating the values (Thompson, 1993), teachers have a potentially small but useful set of guidelines for fostering the sociomathematical norm of *speaking with meaning* (Clark, Moore, & Carlson, 2008) whereby the classroom atmosphere is about making sense of others' thinking and representing one's own thinking. In such an environment, teachers are primed to be more disciplined at noticing and motivated to make use of the thinking they do notice.

Practices that Support Meaning-Making

Our experience with teachers and review of relevant literature suggest that it is difficult for teacher professional development to achieve measureable gains in student learning. Our hypothesis is that a small set of teaching practices focused on using quantitative reasoning to generate meaningful representations of quantitative relationships can foster more focused noticing of student meanings and improve a teacher's ability to react productively to student thinking. Synthesizing the works described in our theoretical perspective, we generated the following set of practices and expectations that we call *meaningful mathematical communication*, or MMC, practices.

1. An expectation that people speak clearly and meaningfully by avoiding the use of pronouns and referencing specific quantities in class conversations and written assignments (Clark et al., 2008; Mason, 2002).
2. An expectation that, when a person writes an expression or performs a calculation, she first writes out in words or says verbally what she intends to calculate. After a person writes an expression or performs a calculation, she justifies what the expression or calculation represents (Thompson, 1993; Thompson, 2013).

3. An expectation that when a person writes a formula or mathematical expression he is attempting to communicate his thinking. Therefore, each person in the course is responsible for being able to justify what his mathematical statements are intended to convey. If someone else creates a different representation (including mathematically equivalent statements that contain different expressions or orders of operations) then it may represent a fundamentally different way of thinking, and each person is responsible for trying to understand this thinking (Thompson 1993; Thompson et al., 1994).

The following example highlights why we think these are important classroom norms to establish and support. Consider an arithmetic series with an even number of terms. Figure 1 summarizes three methods one might use to efficiently determine the sum.

Method 1	(sum of the *n* terms of the series) is (number of pairs) (sum of each pair)	$S_n = \left(\frac{n}{2}\right)\left(a_1 + a_n\right)$
Method 2	(sum of the *n* terms of the series) is (number of terms) (average term value)	$S_n = \left(n\right)\left(\frac{a_1+a_n}{2}\right)$
Method 3	(sum of the *n* terms of the series) is half of ((the number of pairs formed by combining the original series with the same series in reverse order) (sum of each pair))	$S_n = \frac{1}{2}\left[\left(n\right)\left(a_1 + a_n\right)\right]$

Figure 1. Different ways to conceptualize finding the sum of an arithmetic series.

The three formulas $S_n = \left(\frac{n}{2}\right)\left(a_1 + a_n\right)$, $S_n = \left(n\right)\left(\frac{a_1+a_n}{2}\right)$, and $S_n = \frac{1}{2}\left[\left(n\right)\left(a_1 + a_n\right)\right]$ produce the same sum for every arithmetic series. But it is not enough to treat the formulas off-handedly as merely mathematically equivalent statements and settle on one as a preferred method. Each formula derives from a different conception of efficiently calculating the sum and, when treated in this way, helps students understand that mathematical symbols and formulas are ways to communicate and represent their thinking. Explaining what someone intends to calculate before generating the mathematical representation helps to highlight this point. It also promotes synchronizing the reasoning process that generated the formula with the order of operations one performs while evaluating it and highlights how a different order of operations corresponds with a different way of reasoning.

We hypothesize that most teachers can implement the MMC practices regardless of the level they teach or their background in attempting to make student thinking an important part of their lessons. We also believe that the MMC practices are not overly ambitious in terms of the demands they place on teaches.

Research Questions

Based on our theoretical perspective and our conceptualization of the MMC practices, we designed a professional development intervention and conducted a study aimed at answering the following research questions. 1) How did the teacher come to make sense of quantitative reasoning as a lens for interpreting symbols and processes? 2) How, and to what degree, did the teacher attempt to recreate similar experiences for her students? 3) To what extent did the teacher adopt the MMC practices as classroom norms and use them to gather evidence about her students' meanings and reasoning? [In other words, did the teacher engage in *disciplined and productive noticing*? We say a teacher engages in disciplined and productive noticing when she intentionally chooses to gather evidence of student thinking and leverages what she notices to make *productive instructional moves* such as posing questions to help students build new connections, asking students to compare their reasoning to their peers' reasoning, or generate alternative representations of their thinking.]

Methods

This study involved one teacher in a large urban middle school in the Southwestern United States. Tracy was teaching an honors Algebra II course using the *Pathways Algebra II* curriculum (Carlson & O'Bryan, 2014). The study included five one-on-one professional development sessions and five classroom observations spread throughout the Fall 2014 semester and concluded with a final follow-up interview near the end of the Spring 2015 semester. The professional development sessions had two goals. First, to present tasks and contexts where thinking carefully about quantities' values being represented by mathematical expressions is useful and where mathematically equivalent statements can represent different ways of conceptualizing a context. Second, for Tracy to reflect and comment on anything she observed while implementing a similar focus during class sessions with students. We conducted classroom observations to determine if Tracy implemented the MMC practices and, if so, the effect they had on the quality of classroom discourse.

The first author video recorded and transcribed each professional development session and most classroom observations. A delay in receiving signed parental consent forms for videotaping classroom sessions necessitated using shorthand techniques to transcribe the first two classroom sessions in person. He later expanded these notes to a very close approximation of the class conversations. We analyzed the transcripts using open coding (Strauss & Corbin, 1990) and then looked for recurring and emergent themes over the course of the semester. In this paper we report results relative to one identified theme (increased noticing).

Results

Mason (2002) says that a teacher with a disciplined and systematic approach to noticing creates opportunities to gather important feedback from her students, and this feedback is used in planning for and acting in future learning moments. For Excerpt 1, the first author asked Tracy to explain anything that stood out to her while grading a recent quiz.

Excerpt 1.

1 *Tracy:* Um on number three…You leave a school and start walking home at a constant speed of four feet per second. Five minutes later you are a hundred feet from your house. Assume you're walking in a straight path to your house. Write a formula to define the relationship between the quantities distance from your house and time since leaving the school. And so then the first one was writing a formula telling them how to define the variables, and then the second two questions were so how far is your house from the school and how long did it take you to get there. What was interesting is almost everybody got b and c correct. A lot of people missed a. Because they were like I know you told me that the distance should be distance from my house, but what I would really like to do is distance from school. (*laughs*) So a lot of them used an incorrect reference point um or they, they said okay I'll use distance, but I'm gonna make it this other distance. So we really have done several problems like this after that…I've had them walk across the room. [Student] was on the little pretend bicycle …I have problems where I had them model okay so now back up what's happening?

And go forward, what's happening to the distance? And I also did several problems after this where I made them write both formulas. Okay, you really wanted to do this formula, fine do that one and do this one, and so they knew because they had to do two that they had to be different, and then we also talked a lot about that afterwards. How, one thing that I realized they were doing is your reference point has to match your variables, so like they would be using a reference point where the *d* in the reference point was distance from house, or school in this case, so they're using a reference point maybe of distance from school, but then they're trying to write a formula distance from house. Well, if your formula's tracking distance from house, your reference point has to be in distance from house, or if it's distance from school, it has to be in-, so we talked about the importance of, what my, what the meaning of the reference point and the variables that it's keeping track of, those need to be the variables that are appearing in the equation so then that got back to a discussion of and how important is it that we define (*laughs*) our variables accurately.

Tracy noticed a difference between how she and her students conceptualized a situation. This was an important moment for her because she saw a need to practice thinking about different ways to conceptualize a situation and how choosing different pairs of co-varying quantities to compare creates different relationships. After Tracy noticed that students did not conceptualize the situation as she intended, she modified her future instruction to account for this tendency and was primed to notice similar inconsistencies in the future.

In Excerpt 2, Tracy described her students' work to interpret the parts of the general explicit formula for geometric sequences.

5. We can create a general formula that serves as a model for the explicit formulas for *all* geometric sequences.
   a. What is the explicit formula that defines the value of a_n for a geometric sequence with an initial term value a_1 and a common ratio r?
   b. Explain what each of the following represents in the formula (be clear and specific).
      i. n ii. a_n iii. $n-1$ iv. r^{n-1} v. $a_1 \cdot r^{n-1}$

Figure 2. Exploring student meanings for expressions (Carlson & O'Bryan, 2014, p. 94).

Excerpt 2

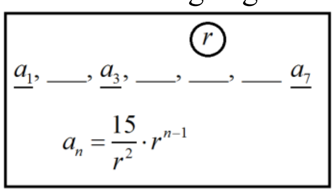
1 *Tracy:* [See Figure 2]. The one that was most concerning to me actually out of these five, cuz I feel like they got most of them, well like-like for example on this one [points to part (b. v.)], they said this is the first term multiplied by the ratio *n* minus one times, and I said, well, what does that find? Well, the n^{th} term. Okay, so this represents the n^{th} term. Then they said this- when I asked them what this was [points to part (b. iii.)] they said that's the position before the n^{th} position. They didn't see it as that's how far you need to change, that's how far your position is changing away from one. So that led to a discussion. And so, and then it had to be linked to this is multiplying by the ratio this many times but why? Because you're changing away that many positions.

According to Tracy, her students had little difficulty generating the formula $a_n = a_1 \cdot r^{n-1}$ inductively from several examples. But because Tracy asked them to describe their meanings for the different parts of the formula, she noticed that their meaning for at least one of the expressions differed from her own (and thus from the meaning she expected them to develop). Tracy said that in the past she was satisfied once students generated and could use the explicit formula, but she would not have realized that the students' conceptualizations of the formula differed from her own. After this experience Tracy created her own set of follow-up tasks designed to help students build a meaning for $n - 1$ as the value of the quantity "change in term position away from $n = 1$".

In the professional development sessions the first author encouraged Tracy to develop a personal appreciation for how different ways of conceptualizing a situation might lead to different (but equivalent) mathematical representations and that this point of view creates flexibility in modeling a situation. During the third such session, Tracy realized that she could use any term in the sequence as a reference point for relating the term values to their positions. That is, instead of $a_n = a_1 \cdot r^{n-1}$ representing the general explicit formula for a geometric sequence, she could think about the general formula as $a_n = a_j \cdot r^{n-j}$ where a_j is the j^{th} term in the sequence. Tracy said that she wanted her students to develop a similar understanding, and so she wrote and assigned the problem shown in Figure 3. She wanted students to understand that they could write the formula as either $a_n = 15 \cdot r^{n-3}$ or $a_n = 143 \cdot r^{n-7}$ with an appropriate choice of r. However, she was also sensitive to noticing alternative valid representations by working to make sense of the thinking behind these representations and wanted to help her students do the same.

> Given a geometric sequence with $a_3 = 15$ and $a_7 = 143$, write the explicit formula. (*Note that you do not need to calculate the value of r*.)

Figure 3. Tracy's problem for defining a geometric sequence's explicit formula.



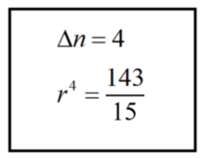


Figure 4 (left). Student response to the warm-up. *Figure 5 (right)*. Tracy's clarification.

Excerpt 3

1 *Tracy:* Yeah. Now take a look at S1's work. I want you to just read her work and think about it. [*Tracy shows Figure 4.*] S1, can you explain your work?

2 *S1:* Well, first we tried to find *r* but it was really messy. So we wanted to just leave it as *r*.

3 *Tracy:* So you did find *r*?

4 *S1:* Yeah. It was about nine point five three three three and then we had to take a root. It was kind of like the second one.

5 *Tracy:* So...wait. Tell me how you got nine point five three?

6 *S1:* The change in position was four, so *r* to the fourth is a hundred forty three divided by fifteen. [*Tracy writes Figure 5.*]…So then we focused on finding the first term, which is two positions away so we divided by *r* squared.

7 *Tracy:* [*long pause*] Should this work? [*pause*] Think back to yesterday, think of the formula we developed and the reference point. What did we use?

8 *S2:* One comma *a* one. The first term.

9 *Tracy:* Yes. And our goal was to write a formula that allows us to calculate the output *a n* for any input *n*. Think about the parts of the function and what they represent. What does *a* one represent?

10 *S2:* First term.

11 *Tracy:* Okay, *a* one is our first term. What does *r* represent?

12 *S3:* What we multiply by to get the next term.

13 *Tracy:* What is *n* minus one? We had some discussion yesterday about the possible interpretations. Some of you said that it was the position before the n^{th} position, but we talked about how that might not be the most useful interpretation here.

14 *S3:* It's the change away from one.

15 *Tracy:* Yes, the change in term position away from one. Now we don't know *a* sub one, instead we know *a* sub three, but by dividing by *r* to the second we can represent the value of *a* sub one even though we don't otherwise know what it is.

In another lesson, Tracy asked students to consider the contribution of a fictitious classmate who supposedly made the claim about finite geometric series shown in Figure 6.

One of your classmates claims that he can calculate $S_6-(3\cdot S_6)$ without knowing the sums of the series. He says that a_1-3a_6 will have the same value as $S_6-(3\cdot S_6)$ and shows the following work. Is he correct?

$$S_6=5+\cancel{15}+\cancel{45}+\cancel{135}+\cancel{405}+\cancel{1,215}$$
$$\underline{-\left(3\cdot S_6= \quad \cancel{15}+\cancel{45}+\cancel{135}+\cancel{405}+\cancel{1,215}+3,645\right)}$$
$$S_6-3\cdot S_6=5-3,645$$
$$S_6-3\cdot S_6=-3,640$$

Figure 6. Exploring a claim about geometric series (Carlson & O'Bryan, 2014, p. 106).

Tracy's goal was for students to see that the difference between the sum of the series and *r* times the sum of the series was the difference between a_1 and $r\cdot a_n$. As students worked on this task, Tracy asked them to think about the meaning of various expressions, and in doing so noticed that some students, seeing $S_6 - 3\cdot S_6 = 5 - 3,645$, were confused because S_6 was not 5 and $3\cdot S_6$ was not 3,645. We believe that a focus on the MMC practices helped Tracy notice and address this incorrect conception and reinforce what the various expressions represented in this context.

Discussion

We believe that teachers who state their learning goals in terms of student thinking (and not just in terms of performance objectives) are better equipped to monitor the development of students' meanings and to respond productively in the moment when they notice that these emerging meanings deviate from intended meanings. Excerpts from Tracy's teaching demonstrate that a teacher can leverage quantitative reasoning to make fine grained observations about students' mathematical reasoning. Specifically, Tracy worked to implement the MMC practices in her classroom on a daily basis and thus developed a disciplined practice of noticing that prompted her to adjust her instructional activities and trajectory to focus on and monitor students' mathematical meanings and make *productive instructional moves*.

We acknowledge that this case study involved only one teacher and that we do not have detailed observations of Tracy's teaching prior to her participation in the study. We only have Tracy's testimonial that the MMC practices were a significant contributing factor in what we witnessed and so we stop short of claiming that our professional development intervention in particular was responsible for Tracy engaging in *disciplined and productive noticing*. However, it was clear that Tracy leveraged quantitative reasoning in reinterpreting her own meanings for algebraic expressions and processes, designed lessons to create similar opportunities for her students, and leveraged quantitative reasoning to interpret and respond to students' classroom contributions. Thus, we believe that quantitative reasoning can serve as a useful tool to help teachers refine their noticing in the context of teaching mathematics. We hope that future research can develop and refine a framework to characterize teachers' quantitative reasoning and the connection between quantitative reasoning and disciplined and productive noticing.

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