Developing an open-ended linear algebra assessment:
Initial findings from clinical interviews

Muhammed Haider, Khalid Bouhjar, Kelly Findley, Ruby Quea, Christine Andrews-Larson
Florida State University

The primary goal of this study was to design and validate a conceptual assessment in an undergraduate linear algebra course. We work toward this goal by conducting semi-structured clinical interviews with 8 undergraduate students who were currently enrolled or had previously taken linear algebra. We try to identify the variety of ways students reasoned about the items with the intent of identifying ways in which the assessment measured or failed to measure students' understanding of the intended topics. Students were interviewed while they completed the assessment and interview data was analyzed by using an analytical tool of concept image and concept definition of Tall and Vinner (1981). We identified two themes in students' reasoning: the first theme involves students reasoning about span in terms of linear combinations of vectors, and the second one involves students struggling to resolve the number of vectors given with the number of entries in each vector.

Key words: assessment, linear algebra, inquiry-oriented, student thinking

Students from a variety of science, technology, engineering, and mathematics (STEM) disciplines are required to take linear algebra as part of their undergraduate mathematics coursework. Students typically struggle with the theoretical nature of linear algebra as it is often their first time grappling with abstract mathematical concepts (Wawro, Sweeney, & Rabin 2011). Students' mathematical background up to this point is often primarily computational in nature; this often creates a barrier for students to overcome when they reach linear algebra (Carlson 1993).

Linear algebra is a pivotal course that includes mathematical underpinning of different STEM fields, but it is rife with challenges for students. According to Wawro (2011), "The content of linear algebra, however, can be highly abstract and formal, in stark contrast to students' previous computationally-oriented coursework. This shift in the nature of the mathematical content being taught can be rather difficult for students to handle smoothly." The abstract concepts of linear algebra are often taught in such a way that students do not find any connections between new linear algebra topics and their previous knowledge of computational mathematics (Carlson 1993). Researchers have worked to address this issue by developing inquiry-oriented instructional materials that help instructors and students bridge students' informal and intuitive ideas with more formal and conventional understandings (Wawro et al., 2013) This work aims to move toward documenting the effectiveness of these materials in supporting students' conceptual understanding of central topics in an introductory undergraduate linear algebra course.

In this study we have designed an assessment that aligns with four focal topics typically covered in an introductory linear algebra course: (1) linear independence and span, (2) linear systems, (3) linear transformations, and (4) eigenvalues and eigenvectors. We aimed to identify two questions for each of these four topics in order to develop an 8-item written assessment that could be completed by students in less than one hour. Based on findings from similar studies, we anticipate that we might see greater conceptual learning gains for students who learned in inquiry-oriented classrooms along with similar procedural learning gains (Rasmussen & Kwon, 2007). Research questions for this proposal are:

- What is the nature of student thinking elicited by the items on our assessment draft?
- To what extent do the items accurately measure student thinking?

Literature & Theoretical Framing

Difficulty in teaching and learning of linear algebra during students' first year of undergraduate study is well documented (Hillel, 2000; Sierpinski, 2000; Stewart & Thomas, 2009). Students often struggle with fundamental concepts like span, linear dependence, linear independence, and basis (Stewart and Thomas, 2009). Additionally, the need to learn and coordinate modes of the description and representation of abstract concepts of linear algebra can function as a source of difficulty for students (Hillel, 2000).

A theoretical construct that has been useful in many areas of mathematics education for making sense of students' struggles as they work to make sense of a new idea is the notion of concept image and concept definition (Tall & Vinner, 1981). The key distinction here is that the ways in which students reason with and about a mathematical construct is often different from (and often at odds with) the definition of that construct which is accepted by the broader mathematical community.

Researchers have been using the constructs of concept image and concept definition to analyze and understand students' thinking and understanding of concepts for more than three decades (Wawro et al. 2011). Britton and Henderson (2009) made use of concept image and concept definition to analyze the conceptual difficulties of students in linear algebra, especially about vector space and subspace. We draw on Tall and Vinner's (1981) notion of concept image and concept definition as an analytic tool for interpreting students' responses to assessment items.

According to Tall and Vinner (1981) concept image is the "total cognitive structure that is associated with the concepts, which include all the mental pictures and associated properties and process" (Tall & Vinner, 1981 p.152). For a given concept, every individual creates an image or structure in their mind that helps the individual understand and remember that concept. This concept image may or may not be similar to other individuals' images, and these images can be quite different from the formal definition of the concept. Moreover, Wawro et al. (2011) contend that concept image is not a static entity; it instead changes over the time and with new knowledge. Tall and Vinner (1981) use the term 'formal concept definition' to refer to the definition that is largely accepted by the mathematical community; they argue that this can be different from an individual's 'personal concept definition,' which may change over the time and with new knowledge as is the case with one's concept image. For our analysis, we look for alignment between a student's elicited concept image and the formal concept definition as evidence of understanding.

Data Sources

In this study, we conducted hour-long semi-structured clinical interviews (Bernard, 1988) with 8 university undergraduate students: 6 males and 2 females. One of the participants was taking linear algebra at the time of the interview, and the other participants had taken linear algebra within the last two years. The participants' majors covered fields that included mathematics, education and economics. Participants had taken an average of four math classes after linear algebra.

Every participant was asked to work through eleven assessment questions using a think-aloud interview protocol, in which the interviewer asked the student to read each item aloud and think aloud as he or she came to an answer. The interviewer then asked follow-up questions as needed to understand the student's reasoning in arriving at their answer. Each interview lasted for approximately one hour and was audio and video recorded. In this preliminary report, we consider participants' responses to the first interview question, shown below in Figure 1.

1. Answer the following questions regarding the set of vectors $V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.
   a. Which of the following best describes the span of the set V?
      i. A point
      ii. Two points
      iii. A line
      iv. Two lines
      v. A plane
      vi. Two planes
      vii. A 3-dimensional space

   b. Give an example of one vector in the span of V, and show or explain how you found that vector.

   c. Give an example of one vector in R^3 that is not in the span of V and explain how you know it is not in the span of V, OR explain why there is no such vector in R^3.

Figure 1: Assessment item focused on span

We developed the assessment items used in this study by consulting past assessments prepared by 5 different mathematics faculty members at different universities, some of whom had been involved in the development of the IOLA materials, and others of whom had not. After identifying a set of questions related to each of our four focal topics, three mathematics faculty members from three different institutes were consulted to identify which items these experts felt focused on key ideas and had the potential to assess students' conceptual understanding of these ideas. We modified our assessment according to experts' initial feedback, and the assessment items to be used in interviews were selected after receiving a second round of feedback from these experts. We piloted the assessment with two students and made minor adjustments based on these students' responses. This modified assessment was used for the remaining interviews.

Methods of Analysis

In order to identify the kinds of student thinking elicited by our assessment items and the extent to which these accurately assessed student understanding, we conducted our analysis in four phases: (I) characterizing individual students' concept images, (II) identifying themes across students, (III) documenting students' written responses, and (IV) relating written responses to concept images elicited. Phases I and II will allow us to identify the kinds of student thinking elicited by our assessment items. Phases III and IV will offer insight to the extent to which the items accurately assessed students understanding. Specifically, we look to see whether the assessment item accurately documents alignment between student's concept image and the formal concept definition. The phases of analysis are described in greater detail below.

Phase I: Characterizing individual students' concept images. We developed a short description of each student's concept image of span by first watching the video and transcribing each student's interview response to question 1. We then developed a list of themes that characterized how he/she thought about span and collected quotes that exemplified characteristics of the student's thinking.

Phase II: Identifying themes across students in how students reason. In this phase, we grouped students according to the nature of their concept images. This helped us document themes in how students reason about the items. These groupings of students' concept images were organized in a table to make it easier to identify trends in thinking.

Phase III: Documenting written responses. In this phase, we identify what students stated their final answer would be (and other answers they offered if they changed their mind) as well as the justification they offered for their answer(s). This was identified by drawing on students' written work as well as using audio/video data as needed in cases when the response was given orally but not written on the student pages.

Phase IV: Aligning concept images with item responses. Students' responses to each item on the assessment were aligned with their corresponding concept image. Each response was color coded to indicate whether a correct response corresponded to correct or incorrect reasoning and whether an incorrect response corresponded to correct or incorrect reasoning. This will be used to assess the extent to which the item accurately measured what we intended.

Findings

After interviewing students and transcribing their interviews we analyzed the first assessment item to document students' concept image of span. In this preliminary report, we summarize themes we noted in students' concept images on this item and speculate on what this tells us about what our item is measuring, as well as what it needs to measure. In our presentation, we will provide a synopsis of all four phases of analysis for this item as well as other items on this assessment.

We identified two themes in students' reasoning as evidenced by their responses: the first theme involves students reasoning about span in terms of linear combinations of vectors, and the second theme involves students struggling to resolve the number of vectors given with the number of entries in each vector. (In addition, there was one student who didn't remember what span was, so he answered all parts of the question as if it was just referring to the set of vectors *V* rather than the span of that set of vectors; using this reasoning the student gave correct answers for 1b and 1c.)

Three of the students reasoned about all parts of the span assessment item in terms of the set of all possible linear combinations of the set of vectors given. Unsurprisingly, these three are the students whose concept image was consistently well-aligned with the formal concept definition. Interestingly, all three of these students offered rich geometric interpretations as part of their elicited concept image. This suggests to us that geometric intuition might be an important aspect of the concept image needed for students to successfully reason through this item (and potentially other items) regarding span, even though the formal concept definition of span does not necessarily entail a geometric interpretation. For example, one student Lewis explained his reasoning to question 1a: "A span is a linear combination or it would be any kind of linear combination of these two [pointing towards V]… because they are linearly independent, so any span of these two vectors will be linear combination of the two vectors so reproduce a plane."

Four students struggled to resolve the number of vectors given with the number of entries in each vector, but they resolved this issue in a variety of ways. For instance, one student noted the vectors were in three dimensions and concluded (incorrectly) that the span must be 3-dimensional, meaning that no vector in R^3 can be outside of the span. Another student, Beth, struggled with the same issue, but resolved it correctly by reasoning that "each vector has three entries in its column … that means that it is in the third dimension, I think there is only two vectors though, I think I need a third vector in order for this to actually span the third dimension and so since there are two it will span just the second dimension and the second dimension will be a plane so then it might actually just be a plane." One student resolved the issue by putting the vectors of the matrix *V* into a matrix, row reducing the matrix, and counting the number of pivot columns. Since there were two of something, he felt the span should be either two points or two lines, but he wasn't sure which because he didn't have a geometric interpretation. The final student concluded that the span of *V* would be two planes because each vector represents a plane. Interestingly, these students tended to give a linear combination of the vectors of *V* on part b of the question when asked for an example of a vector that was in the span (though some had interesting limitations on how those combinations should be formed, e.g. thinking the coefficients had to be integers). This suggests to us that a significant source of difficulty for students developing a rich concept image of span lies in coming to think simultaneously about *all* possible linear combinations of a set of vectors.

Implications/Future Work

Our findings suggest two things about the design of an assessment item focused on documenting students' understanding of span. First, is that we need to find a way to assess students' strategies for resolving differences between the number of vectors and the number of entries in each vector. Second, we likely want to include separate prompts that offer insight into students understanding of linear combinations and their understanding of span as the set of *all* possible linear combinations. We have endeavored to assess the latter using a geometric approach. Extending our analytic strategy, we intend for our analysis of other items to similarly inform aspects of student understanding that need to be measured by our assessment.

Questions for Audience

- When we conduct quantitative analysis, how do we account for the relatedness of subparts of questions (e.g. 1a, 1b, 1c)? We view this as a strength of the assessment, but are unsure of how to account for it methodologically.
- What is the contribution of this work? Is it methodological (is the method of refining assessment items new/novel/worth writing about)?
- How can we think about assessing the quality of the assessment as a whole rather than item by item?

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