Students’ conceptualizations and representations of how two quantities change together

Kristin M. Frank
Arizona State University

In this article I discuss the nature of two university precalculus students’ meanings for functions and graphs. I focus on the ways in which these meanings influence how these students reasoned about and represented how two quantities change together. My analysis revealed that a student who views a graph as a static shape and does not see a graph as a representation of how two quantities change together will not be successful in constructing meaningful graphs, even in instances when she is able to reason about two quantities changing together. Students made progress in seeing graphs as emergent representations of how two quantities change together when they conceptualized the point (x,y) as a multiplicative object that represented the relationship between an x and y value.

Key words: Function; Covariational Reasoning; Graphing

There is a growing body of research that documents the importance of covariational reasoning, imagining quantities’ values varying together, when conceptualizing rates (Johnson, 2015; Thompson, 1994a; Thompson & Thompson, 1992), the behavior of exponential and trigonometric functions (Castillo-Garsow, 2010; Moore, 2010; Thompson, 1994c), and graphs (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Moore & Thompson, 2015). After high school, reasoning about variation is essential to understand calculus (Thompson, 1994b; Zandieh, 2000), differential equations (Rasmussen, 2001), and continuous functions (Roh & Lee, 2011). While the research community understands the importance of covariational reasoning, researchers do not yet understand how students come to reason covariationally.

Carlson et al. (2002) developed a framework to classify student’s covariational thinking but they did not describe how students might develop these ways of thinking. Whitmire (2014) studied how undergraduate students solve a task whose solution necessitated covariational reasoning. He found that static shape thinking, reasoning about a graph based on one’s perceptions of the shape of the graph (Moore & Thompson, 2015), stood as a distraction from covariational thinking. In contrast, he found that simultaneously attending to two quantities was associated with instances of covariational reasoning. In this report I extend Whitmire’s findings by elaborating the ways static shape thinking inhibits reasoning covariationally and I provide evidence that simultaneously attending to two quantities’ values is propitious for engaging in covariational reasoning and representing how two quantities’ values change together.

Images of Variation

To engage in covariational reasoning, one must construct an image of how two quantities’ values change together. As Saldanha and Thompson (1998) described, this requires the student construct a multiplicative object and conceptualize the two quantities’ values at once. Then the student “tracks either quantity with the immediate, explicit, and persistent realization that, at every moment the other quantity also has a value” (Saldanha & Thompson, 1998, p. 2). For example, in the context of graphing, the student must construct the point (x, y) as a multiplicative object that simultaneously represents both the value of x and y. Then the student can track (or imagine tracking) the value of x with the awareness that as x varies, y varies as well.

Castillo-Garsow (2010) described two ways for students to imagine tracking the value of x: (1) the student imagines the tracking already happened and conceptualizes a completed change in the value of x – a chunky image of variation or (2) the student imagines change in progress and conceptualizes sweeping over a continuum of x-values – a smooth image of variation. Lakoff and Núñez (2000) argued that conceptualizing sweeping over a continuum involves *fictive motion* – using a motion verb when the subject is not actually moving. For example, in the phrase "the value of x goes from 1 to 4" the value of x is not moving but we talk as if it is. Fictive motion enables one to go between static and dynamic conceptualizations of the value of x.

Tracking a quantity's value is a nontrivial activity for students. This necessitates that the student conceptualize a quantity, the measure of that quantity, and that measure varying in a situation. If the student does not construct this image the student is said to have no image of variation.

Methodology

I conducted one-on-one task-based interviews with three university precalculus students, Sara, Carly, and Vince. The students were selected from three different sections of precalculus to account for differences in instruction. The interview consisted of two phases. The first phase was a clinical interview (Clement, 2000; Hunting, 1997). I engaged the students in tasks I anticipated would support me in understanding their meanings of functions, tabular representations, and graphical representations. The second phase of the interview was a task-based-teaching interview (Castillo-Garsow, 2010; Moore, 2010). My primary teaching goal was to support students in conceptualizing a graph as an emergent representation of how two quantities change together. In this part of the interview I used dynamic animations to support students in conceptualizing *change in progress*. I anticipated this would support students in imagining sweeping over a continuum of values thus constructing an image of smooth variation.

Results

After I completed the interview process I engaged in grounded coding (Strauss & Corbin, 1998). After each interview I engaged in open coding and gathered evidence of how students conceptualized functions, graphs, and variation. After reviewing videos and transcripts of each interview I found that while all three students were able to describe how two quantities changed together, not all students were able to represent their conceptualization of how two quantities changed together. Since Sara and Carly exhibited similar ways of thinking, I will focus on contrasting Sara and Vince's ways of thinking.

The Story of Sara

Over the course of two interviews, Sara consistently used shape thinking and memorized procedures to make sense of problem situations. For example, at the beginning of my interview with Sara I asked her to explain what it means for something to be a function. She responded by describing a procedure to compute the change in the value of y and divide it by the change in the value of x. She did not discuss the meaning for these calculations nor did she explain the meaning of the result of dividing.[1] Her responses throughout the interview suggest that she was unable to view graphs or other function representations as a means of representing how two varying quantities change in tandem. Data to support this claim follows in the next section.

Sara associated graphs with shapes she had previously seen in math class

I presented Sara with a graph that appeared to show two vertical lines, one at $x = 1$ and the other at $x = -1$ (Figure 1a). I asked her to determine whether the graph represented a function. She gestured that it was like an upside down 'u' shape and explained, "only you can't see the top". She concluded that the graph represented a function since the graph was like one she had seen in class. She matched the shape with one she had seen before and justified her response based on her perception of the graph's shape. This suggests that for Sara, graphs represent shapes as opposed representations of how quantities' values vary together.

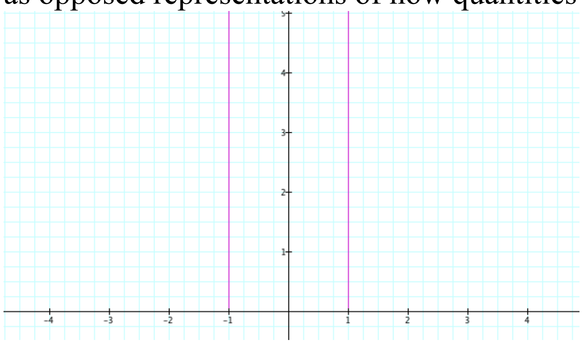

Figure 1a: Sara was asked to determine whether this relationship is a function.

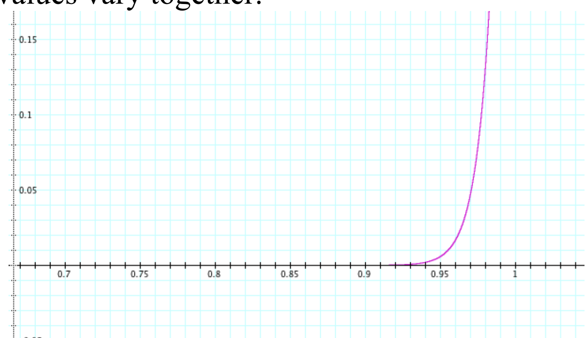

Figure 1b: Graph displayed to Sara after she anticipated the behavior around at $x = 1$.

I asked Sara to anticipate what we would see if we zoomed in on the graph around the point (1, 0). She described that the line would come down perpendicular to the horizontal axis but just stop and not go below the horizontal axis. Then I highlighted a region on the graph around $x = 1$ and zoomed in on this region (Figure 1b). I asked Sara if she believed what she saw and she said,

> "No. (Laughs) I mean I haven't seen it before and I just feel like a graph would look weird if it is like going down and then curving also. If like. Especially if there is a top to graph, which I don't know if there is or not now because I am second-guessing myself. But. (4 seconds of silence) Or, there is and there are asymptotes there. There wouldn't be a top. So that could work if there was like two asymptotes."

She proceeded to sketch of her new understanding of the graph (Figure 2)

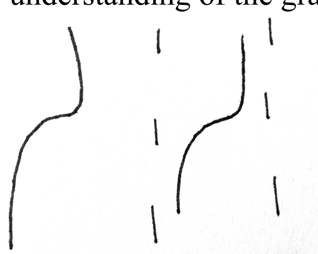

Figure 2. Sara's graph after she conceptualized graph as a shape with asymptotes.

[1] It is noteworthy that Sara had recently learned a method for determining the average rate of change of a function on an interval of a function's domain.

Sara approached this task by trying to think of a graph she had seen before that matched her conception of the situation. When her prediction did not match the graph I displayed, she used the new information displayed in the graph to develop a new possibility for the shape of the graph – a graph with asymptotes. Instead of conceptualizing what the zoomed in graph told her about how x and y varied together, Sara concluded that the graph would look weird "going down and curving". This provides further evidence that Sara engaged in static shape thinking when making sense of graphical representations.

Implications of Sara's tendency to engage in static shape thinking

The last task in the interview was based on an item from a diagnostic instrument for professional development (Thompson, 2011). I presented Sara with an animation where the value of Quantity A was represented with a red bar along the horizontal axis and the value of Quantity B was represented with a blue bar along the vertical axis. As the animation played the lengths of the red and blue bars changed together to describe how the two quantities change together. I presented Sara with three versions of this task.

While I intended this to be a novel task, Sara had done a similar task in her precalculus class. Sara said she was bad at these types of problems but her ability complete the first version of this task with ease suggests that Sara learned a strategy to complete these problems. Her way of thinking broke down in the second version of the task which represented the behavior of $y = \sin(x) : -2\pi < x < 2\pi$. As Sara watched the animation, she appropriately described, "As x was increasing at the beginning y was decreasing. But as x comes closer to 0 y also approaches 0 and they both increase for a little bit and as y keeps increasing or as x keeps increasing y starts to decrease." This suggests that Sara was imagining change in progress. Although Sara was able to describe the how the quantities' changed together, she struggled to represent this graphically. In the following excerpt Sara explained her approach to constructing a graph from the animation.

Sara: In my head I like know like as that one is increasing you have to like. I try and like think of the shape of the line or the point or whatever to get to the line. Or yeah.
Interviewer: What do you mean you think of the shape to get to the line?
Sara: So like for this like I have to see how like. Since that [value of x] is like increasing (gestures left to right) and that [value of y] is decreasing (gestures up and down) like what I am thinking in my head. Like I am like trying to figure out which way it needs to go.
Interviewer: Which way what needs to go? The graph?
Sara: Yeah. So. I don't know. That is why it takes me so long when I am just staring at the graphs.

Sara appeared to abandon her thinking about changing quantities when constructing a graph. Instead of conceptualizing the graph as a trace of how the value of x and y change together, she broke the graph up into chunks based on whether the value of y increased or decreased. Then she determined a shape that depicted the appropriate behavior of y as x increased. For example, if the value of y increased as the value of x increased then she knew the graph had to go up and to the right. While Sara was able to appropriately describe how the values of x and y changed together, her tendency to engage in static shape thinking prevented her from leveraging her reasoning about how the two quantities were changing together to construct a meaningful graph.

The Story of Vince

At the beginning of my interview with Vince I asked him to construct a graph from a table of values. He explained that since the table was a function the graph would be a smooth line and he could sketch an approximation but would need the function, "the y equals something", to determine the exact graph. He elaborated that he needed the formula so he could plug in all of the x values, determine the associated y values, and plot all of the resulting (x, y) pairs. This understanding of graphs enabled Vince to draw smooth curves, "a bunch of dots put together that now looks like a line to me." This suggests that Vince conceptualized graphs as collections of points where a point simultaneously represented an x and y value.

While there are limitations to this understanding of functions and graphs, Vince was able to complete all tasks in the interview with this way of thinking. For example, since Vince conceptualized graphs as a collection of points, whenever he attended to two points he also addressed the many points in-between. Specifically, when sketching a graph from a table of values Vince acknowledged that "anything could happen between the given points". Thus, even though Vince often attended to individual points Vince was likely imagining the quantity's value varying. This image of variation seemed to enable Vince to conceptualize sweeping over a continuum of x values with the awareness that he can construct and (x, y) pair at every value of x. This allowed Vince to imagine that "anything could happen" between two values of x.

Leveraging one's image of a correspondence point to construct meaningful graphs

Vince's point-wise meaning for functions and graphs broke down at the end of his interview. The last task Vince completed was the same task that Sara completed (described above). I presented Vince with and animation where the value of Quantity A was represented with a red bar along the horizontal axis and the value of Quantity B was represented with a blue bar along the vertical axis. As the animation played the lengths of the red and blue bars changed together to describe how the two quantities change together (Thompson, 2011). I presented Vince with three versions of this task and each time asked him to sketch a graph of how the two quantities changed together. The following exchange occurred at the beginning of this task. (The animation he was watching during this exchange represented the behavior of $y = 0.2x^3$: $-12 < x < 12$.)

> **Interviewer**: So a little bit different. I have a red line that represents that value of x and the blue line represents the value of y. These two values change together. (Animation plays). Well the question is how do they change together? What if I wanted a graph that showed how these two quantities change together?
> **Vince**: So are you asking for like. Like um. The intersection of the two?
> **Interviewer**: So x is changing and y is changing. Suppose you have a friend in Australia and you can't send them videos. And for some reason you need to tell your friend how x and y are changing – you need to tell him what is going on in this video. So what are you going to do? Anything you could convey in snail mail.
> **Vince**: Um. I would probably do. I don't know. (Gestures two vertical lines). How they are changing? You are no just looking for? I mean I would probably. Hm. If I want him to see the graph. What I am imagining what is happening is there are points. Like you intersect the two and there would be a line – a series of dots. (Gestures smooth curves)
> **Interviewer**: What would that line look like?
> **Vince**: (Gestures curved shape and then draws an appropriate graph).

After three minutes of puzzling about the task, Vince was able to successfully represent the behavior in the animation in the plane. He described a point that he imagined as the "intersection" of the red and the blue line segments and he described keeping track of this intersection as the animation played (Figure 3). Although Vince described this approach at the very beginning of the excerpt, it took more than three minutes of reasoning for him to believe that this would represent how the values of *x* and *y* changed together.

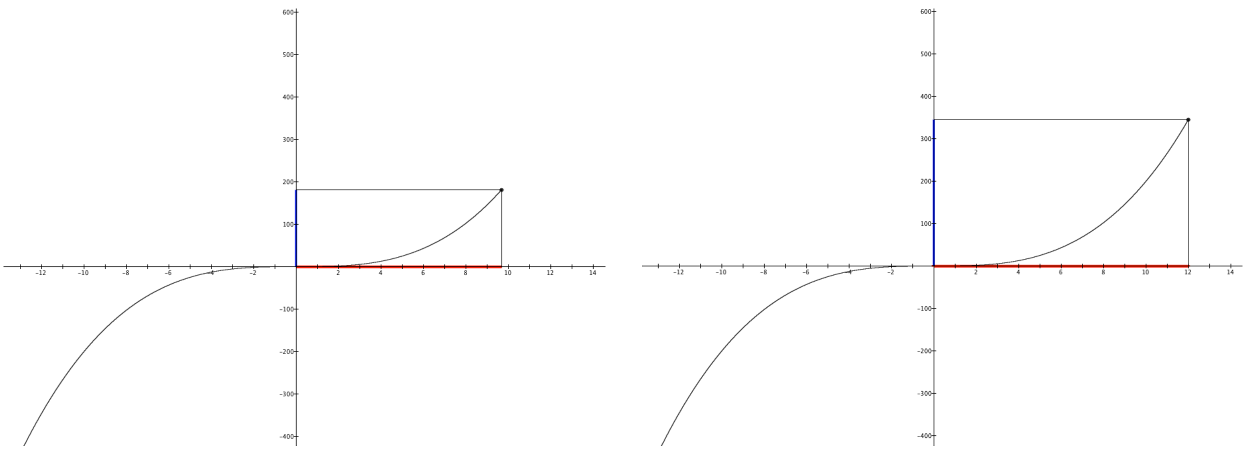

Figure 3: Researcher's representation of the imagery Vince described while constructing a graph to represent the behavior of two continuously varying quantities.

After Vince constructed his graph, I asked him to explain his approach. He began by describing the changes he was conceptualizing:

> "The red is changing at a pretty consistent speed. The blue when it is a lot further down moves a lot quicker then it goes slower and then it starts moving quicker again. So like as. So that is why I thought of this (points to graph in Quadrant 4) because we move up here, you get less bang for your buck. As *x* increases by one the increase of *y* becomes less and less."

Vince operationalized his image of the red line changing at a consistent speed by conceptualizing equal changes in the value of *x*. This gave him a means to think about how the blue line was changing: for a consistent change in *x* the corresponding change in the value of *y* was getting smaller (in magnitude). He was able to use this imagery to confirm his graph appropriately represented the behavior in the animation.

Throughout Vince's interview he exhibited two different tendencies. The first was to describe a continuum of values by attending to all the points in-between two given points and constructing graphs by tracking a correspondence point. His other tendency was to focus on points, numerical values, and calculated changes in the value of a quantity. One possible explanation for these different ways of thinking is to juxtapose Vince's daily experiences with his mathematical experiences. In his day-to-day life, Vince engages with and represents continuous motion. However, in his mathematical experiences he focuses on numerical values, points, and calculated changes in the value of *x* and *y*. As a result of his daily experiences he developed an ability to use fictive motion to construct dynamic conceptualizations of otherwise static objects. He engaged his understanding of fictive motion when he first completed this final task. However, he had little to no experience coordinating fictive motion with mathematics. Thus, he used his understandings from classroom experiences to justify the graph he created.

Vince was able to successfully complete all the tasks in the interview, including representing the relationship between two continuously varying quantities. He was successful because he consistently imagined a quantity's value varying over some continuum and he conceptualized a point as a multiplicative object that simultaneously represented an x value and associated y value.

Conclusion

Vince consistently described graphs as a collection of points really close together. Although Vince experienced some cognitive conflict when trying to graphically represent two continuously changing quantities, he was ultimately successful because (1) he imagined a continuum of x-values, and (2) he had conceptualized a point on the graph as a multiplicative object that simultaneously represented an x value and associated y value.

Sara, on the other hand, constructed meanings for function and graphical representations based on memorized procedures and shapes. While Sarah's conception of a graph as a shape led to many seemingly inconsistent answers throughout the interview, static shape thinking did not prevent Sara from conceptualizing a quantity's value varying continuously and describing how two quantities changed together. Static shape thinking did impact Sara's ability to construct a graph as an emergent representation of how two covarying quantities change together. This suggests that conceptualizing a graph as an emergent trace of two quantities' values requires more than imagining smooth variation and conceptualizing how two quantities change together.

This finding is consistent with Moore and Thompson's (2015) explanation of emergent shape thinking. They explain, "Emergent shape thinking involves understanding a graph simultaneously as what is made (trace) and how it is made (covariation)" (p. 4). This suggests that conceiving covariation is only one aspect of understanding a graph as an emergent representation of how two quantities change together. Vince's ability to conceptualize a graph as an emergent representation suggests that the other aspect of emergent shape thinking entails constructing multiplicative objects. First the student must construct the point (x,y) as a multiplicative object that unites the value of x and the value of y. Then the student must construct the graph as a multiplicative object that unites a way of representing the spatial movement of the multiplicative object with a conception of of covarying values of two quantities.

This study suggests that there are different ways to engage in covariational reasoning. The multiplicative object the student constructs determines the nature of his covariational reasoning. For example, one must construct the point (x,y) as a multiplicative object in order to conceptualize a graph as an emergent trace of how two quantities change together. However, Sara's interview provides evidence that this construction is not necessary in order to describe how two quantities change together. Future studies should explore how to support students in conceptualizing both the point (x, y) and the graph as multiplicative objects. Students have developed robust coping mechanisms that enable them to understand and complete novel tasks using their existing ways of thinking. Thus, educators will need to use unconventional representations, such as leveraging fictive motion, to support students in developing more robust ways of engaging in covariational reasoning and new meanings for graphical representations.

References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic vents: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378. doi: 10.2307/4149958

Castillo-Garsow, C. C. (2010). *Teaching the Verhulst Model: A teaching experiment in covariational reasoning and exponential growth.* (PhD Dissertation), Arizona State University.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In R. Lesh & A. E. Kelly (Eds.), *Handbook of research methodologies for science and mathematics education* (pp. 341-385). Hillsdale, NJ: Erlbaum.

Hunting, R. P. (1997). Clinical interview methods in mathematics education research and practice. *Journal of Mathematical Behavior, 16*(2), 145-165.

Johnson, H. L. (2015). Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning, 17*, 64-90.

Lakoff, G., & Núñez, R. E. (2000). Calculus without space or motion: Weierstrass's Metaphorical Masterpiece *Where mathematics comes from: How the embodied mind brings mathematics into being* (pp. 306-324). New York, NY: Basic Books.

Moore, K. C. (2010). *The role of quantitative reasoning in precalculus students learning central concepts of trigonometry.* (PhD), Arizona State University.

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly (Ed.), *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education*. Pittsburgh, PA: RUME.

Rasmussen, C. (2001). New directions in differential equations: A framework for interpreting sudents' understandings and difficulties. *Journal of Mathematical Behavior, 20*, 55-87.

Roh, K. H., & Lee, Y. H. (2011). The Mayan Activity: A way of teaching multiple quantifiactions in logical contexts. *PRIMUS, 21*(8), 685-698.

Saldanha, L., & Thompson, P. W. (1998). Re-thinking covariation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensah & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education - North America*. Raleigh, NC: North Carolina State University.

Strauss, A., & Corbin, J. (1998). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory*: SAGE Publications.

Thompson, P. W. (1994a). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: SUNY Press.

Thompson, P. W. (1994b). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics, 26*(2-3), 229-274.

Thompson, P. W. (1994c). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld & J. J. Kaput (Eds.), *Research in Collegiate Mathematics Education, 1* (Vol. 1, pp. 21-44). Providence, RI: American Mathematical Society.

Thompson, P. W. (2011). Project Aspire: Defining and Assessing Mathematical Knowledge for Teaching Mathematics: NSF Grant No MSP-1050595.

Thompson, P. W., & Thompson, A. G. (1992). Images of Rate *Paper presented at the Annual Meeting of the American Educational Research Association* (pp. 20-25). San Francisco, CA.

Whitmire, B. (2014). *Undergraduate students' development of covariational reasoning.* (Undergraduate Thesis), Arizona State University.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. Schoenfeld & J. Kaput (Eds.), *Research in collegiate mathematics education, IV* (Vol. 8, pp. 103-127). Providence, RI: American Mathematical Society.