Framework for Mathematical Understanding for Secondary Teaching: A Mathematical Activity perspective

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Abstract: *A framework for mathematical understanding for secondary teaching was developed from analysis of the mathematics in classroom events. The Mathematical Activity perspective describes the mathematical actions that characterize the nature of the mathematical understanding that secondary teachers could productively use.*

Mathematics teaching at the collegiate level focuses on enabling students to develop solid understanding of mathematics. Although collegiate mathematics students often describe mathematics as learning specific topics and strategies and applying this knowledge to their work, their instructors may have additional but less explicit goals such as valuing the structure of mathematics, being able to create a deductive argument, or exploring and comparing systems of mathematics. These latter goals are especially important for prospective teachers of secondary mathematics, and college mathematics instructors are attending in new ways to the mathematical preparation of those who will teach mathematics.

Over the past three decades, mathematics education researchers and theorists have increased their focus on the mathematical knowledge of teachers that helps teachers reach their goals of promoting a more robust understanding of mathematics in their students. During that time, researchers have refined the focus from Shulman's (1986) construct of pedagogical content knowledge to constructs such as mathematical knowledge for teaching (MKT) (Ball, 2003; Ball & Bass, 2003; Ball & Sleep, 2007a; Ball & Sleep, 2007b; Ball, Thames, & Phelps, 2008) and knowledge of algebra for teaching (KAT) (Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2005; McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012). Work on MKT is, perhaps, the best known of the research programs focused on teachers' mathematical knowledge. MKT originated with a reflection on the mathematical knowledge involved in the mathematical work of teaching at the elementary level. MKT partitions the territory of mathematical thinking into categories such as specialized content knowledge, common mathematical knowledge, and mathematics at the horizon. While the MKT categories can partition mathematical knowledge at the secondary level as well as at the elementary level, those categories do not characterize the nature of mathematical thinking that seems to distinguish mathematics at the secondary school level.

In their work in secondary mathematics, students expand their mathematical knowledge to include new ideas such as irrational numbers, complex numbers, static and rotating objects, sample spaces, and a variety of ways to represent these ideas. But the differences between mathematics at the elementary and secondary levels are not solely extensions of the topics involved, but also a change in the nature of mathematical thinking involved. Whereas both elementary and secondary mathematics honor deductive reasoning, secondary mathematics places a much stronger emphasis on deductive thinking within a closed mathematical system. It is in the context of secondary mathematics that curricula focus on reasoning on the basis of a well-defined system of given properties and relationships. For example, the work of secondary students in the study of geometry is more likely to occur at the third or fourth van Hiele level (making deductive connections and constructing proofs) rather than the first or second levels (focused on visualizing or recognizing properties of geometric objects) that are more prominent at the elementary level. At the elementary level, students develop ways to represent mathematical relationships. As students progress through school mathematics, their repertoires of ways to represent mathematical relationships expands so that, as they engage in secondary mathematics, they can be expected to link representations of the same mathematical entities and to reason about a mathematical entity in one representation making conclusions about that entity in another representation.

Secondary teachers need to be able to reason flexibly enough to recognize and act on opportunities for their students to build capacities for reasoning in a closed system and for capitalizing appropriately on a range of representations. They need mathematical understanding that enables them to perform such activities as creating examples, nonexamples, and counterexamples of entities encountered in secondary mathematics, to identify special cases of broad categories of mathematical objects, and to explain when a general statement can or cannot be extended to a larger or different domain or set of mathematical objects. Secondary teachers need to make connections between mathematical systems. In order to facilitate learning secondary mathematics, the work or context of teaching requires a depth of specific mathematical understanding that incorporates the more subtle but important goals of mathematics teaching. Mathematics teachers must not only understand mathematics but they must enable others to understand mathematics in the fullest sense. They need to pose interesting questions and tasks that bring the structure of mathematical systems alive. They need to understand the mathematical thinking of students in order to correct or challenge their thinking. They need to be able to reflect on the curriculum and organization of mathematical ideas. The context of learning mathematics requires specific mathematical understanding beyond pedagogical knowledge.

The six faculty involved (G. Blume, J. Kilpatrick, J. Wilson, and R. M. Zbiek, in addition to the authors) wanted to build a framework that would account for the proficiencies, actions, and work of secondary mathematics teachers. We committed to developing a framework that accounted for the mathematical opportunities secondary teachers actually encounter, and so we began in the classroom. As we began to study the mathematical opportunities unfolding in the classroom, we recognized many of the ideas expressed by others who have attended to secondary mathematics (e.g., Adler & Davis, 2006; Cuoco, 2001; Cuoco, Goldenberg, & Mark, 1996; Even, 1990; McEwen & Bull, 1991; Peressini, Borko, Romagnano, Knuth, & Willis-Yorker, 2004; Tattoo et al., 2008). While the framework incorporates previous ideas, it attends directly to the secondary mathematics built on data from mathematics classes.

Our source of data was a set of what we came to call Situations. A Situation is a mathematical description, based on an actual event that occurred in the practice of teaching, of the mathematics that teachers could productively use in the work of teaching mathematics. Teams of mathematics education faculty at Penn State and at University of Georgia worked with dozens of doctoral students in mathematics education to develop more than 50 Situations. Although any one Situation is too large to report in this paper, we provide a brief outline of one of the Situations (from Heid & Wilson, in press) here. Each Situation includes a Prompt (a description of a mathematical opportunity–an event that one of the authors observed happening in the course of teachers planning or implementing a secondary mathematics lesson) and several Mathematical Foci (development of mathematics that a teacher could productively use in the context of that mathematical opportunity). A short statement about the nature of the mathematical understanding being targeted precedes each Mathematical Focus. Other parts of each Situation are Commentaries (a description of how the Mathematical Foci for the Situation fit together) and PostCommentaries. One of the Situations is outlined in Figure 1.

CHAPTER 22. INVERSE TRIGONOMETRIC FUNCTIONS

Prompt
Three prospective teachers planned a unit of trigonometry as part of their work in a methods course on the teaching and learning of secondary mathematics. They developed a plan in which high school students would first encounter what the prospective teachers called *the three basic trig functions*: sine, cosine, and tangent. The prospective teachers indicated in their plan that students next would work with "the inverse functions," which they identified as secant, cosecant, and cotangent.
Commentary
The Foci draw on the general concept of inverse and its multiple uses in school mathematics. Key ideas related to the inverse are the operation involved, the set of elements on which the operation is defined, and the identity element given this operation and set of elements. The crux of the issue raised by the Prompt lies in the use of the term *inverse* with both functions and operations.
Mathematical Focus 1
An inverse requires three entities: a set, a binary operation on that set, and an identity element given that operation and set of elements.
Secondary mathematics involves work with many different contexts for inverses. For example, opposites are additive inverses defined for real numbers and with additive identity of 0, and reciprocals are multiplicative inverses defined for nonzero real numbers and with multiplicative identity of 1. [Discussion follows about the nature of inverses, the role of an identity in inverses, and the importance of domain and range in consideration of inverses.]
…
Mathematical Focus 2
Although the inverse under multiplication is not the same as the inverse under function composition, the same notation, the superscript -1, is used for both.
[Discussion follows about notation used in different inverse relationships, and the specific use of that notation in consideration of trigonometric functions.]
…
Mathematical Focus 3
When functions are graphed in an xy*-coordinate system with* y *as a function of* x*, these graphs are reflections) in the line* y = x *of their inverses' graphs (under composition).*
The graph of a function reflected in the line $y = x$ is the graph of its inverse, although without restricting to principal values, the inverse may not be a function. Justifying this claim requires establishing that the reflection of an arbitrary point (a, b) in the line $y = x$ is the point (b, a). [A geometric proof follows, using a coordinate plane representation of the reflection of a point (a,b) over the line $y = x$.]
…

Figure 1. Outline describing a Situation appearing in (Zbiek et al., in press).

The Situations we (the cross-university teams) developed suggested a range of mathematical abilities, actions, and settings that could underlie potentially productive mathematical thinking on the part of the teacher. It was on the basis of these abilities, actions, and settings that we embarked on the challenging task of developing our Framework for Mathematical Understanding for Secondary Teaching. As we examined the Situations we had created, we recognized that we needed several different perspectives to explain the mathematics we had identified. Akin to Plato's allegory of the cave, the framework on which we settled consisted of three perspectives, each of which cast a different shadow representing a student's mathematical understanding (See Figure 2).

From one perspective, Mathematical Proficiency, we could use the strands of proficiency to describe the nature of the mathematical understanding, but this perspective did not account for the mathematical actions that secondary teachers could productively take. The second perspective addressed this as Mathematical Activity. However, neither the first nor second perspective accounted for the settings in which teachers needed to call on their mathematical knowledge. The third perspective, Mathematical Context of Teaching, addressed the mathematical context in which teachers could productively call upon their mathematical knowledge.

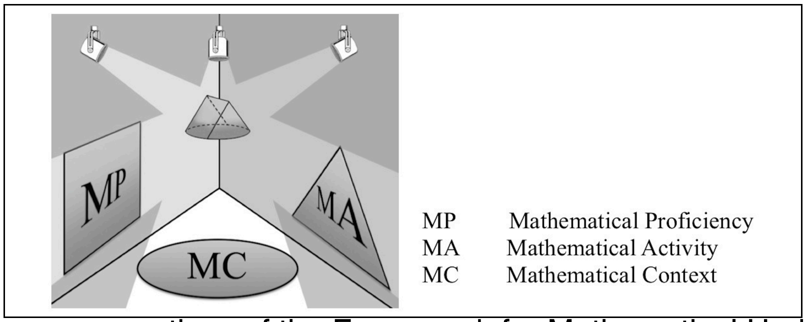

Figure 2. Three perspectives of the Framework for Mathematical Understanding for Secondary Teaching (Heid & Wilson, in press).

The first perspective, Mathematical Proficiency, is likely to be familiar as a way to think about students' mathematical capability. The third perspective, Mathematical Context, provides a description of the mathematical understanding that is particularly relevant to teaching. This perspective was more implicit than explicit in our data, but we realized that the Mathematical Context of teaching indicates why it is critical to recognize and attend to the importance of Mathematical Activity. In this paper, we confine our discussion to the development of the second perspective, Mathematical Activity.

Mathematical Activity

We used the final set of Mathematical Foci as data from which to generate our Framework for Mathematical Understanding for Secondary Teaching. First we identified mathematical actions implicit or explicit in each of the Foci. We then categorized those actions, including categories such as creating mathematical entities and interpreting mathematical representations and orchestrating movement among them.

For example, one set of mathematical actions that we grouped into a single category included the following actions:

- Creating a counterexample for a given structure, constraint, or property
- Creating an example or non-example for a given structure, constraint, or property
- Creating equivalent equations to reveal information
- Creating problems to foreground a concept
- Creating a file (a computer application) whose creation requires mathematics beyond what the file is used to teach
- Constructing an object given a set of mathematical constraints
- Generating specific examples from an abstract idea
- Creating a representation for a mathematical object with known structure, constraints, or properties

Having grouped these actions into a single category, we developed a description of a mathematical action that encompassed these actions. In this case our description was "Creating a mathematical entity or setting from known (to the one creating) structure, constraints, or properties." An example of a specific mathematical action that might fit this category is the task of constructing a quadrilateral with specific characteristics. Other mathematical actions were developed in a similar fashion. A few of the final set of mathematical actions at this juncture, along with specific examples drawn from the Situations, are shown in Figure 3.

Category	Example
Create: Creating a mathematical entity or setting from known (to the one creating) structure, constraints, or properties	Sketch quadrilateral ABCD with $m\angle D = m\angle A = 90$ and $\overline{AB} \parallel \overline{DC}$ such that ABCD is not a parallelogram.
Recognize: Recognizing mathematical properties, constraints, or structure in a given mathematical entity or setting, or across instances of a mathematical entity	Recognizing that strategic choices for pairwise groupings of numbers are critical to one way of developing the general formula for summing the first *n* natural numbers
Choose: Considering and selecting from among known (to the one choosing) mathematical entities or settings based on known (to the one choosing) mathematical criteria	The mathematical meaning of a/b (with $b \neq 0$)) arises in different mathematical settings, including: slope of a line, direct proportion, Cartesian product, factor pairs, and area of rectangles. One might choose slope of a line as a setting to illustrate the need for $b \neq 0$.
Use representations: For given representations, interpret them in the context of the signified, orchestrate movements between them, and craft analogies to describe the representations, objects, and relationships	Using tabular and graphical representations to estimate the value of $2^{2.5}$
Assess (interpret and adapt) the mathematics of the situation: Interpret and/or change certain mathematical conditions/ constraints relevant to a mathematical activit*y*	Assess and use a modulus definition of absolute value in evaluating $f(x) = \left\lvert \sqrt{x-10} \right\rvert$
Extend: Extend the domain, argument, or class or objects for which a mathematical statement is/remains valid.	Extending: the absolute value function from the real to the complex domain; "triangle" from Euclidean to spherical geometry
Connect: By recognizing structural similarity, make connections between: representations of the same mathematical object; different methods for solving a problem; mathematical objects of different classes; objects in different systems; or properties of an object in a different system.	Identifying structural similarities of the Euclidean algorithm and the long division algorithm

Figure 3. A few of the set of mathematical actions that comprised the Mathematical Activity perspective of the Framework for Mathematical Understanding for Secondary Teaching, along with specific examples drawn from the Situations.

Reason: Reason about a mathematical entity in more than one way, including, but not limited to: from mathematical definitions, from given conditionals, from and toward abstractions, by continuity, by analogy, and by using structurally equivalent statements.	Reasoning about the sum of the first n natural numbers by appealing to cases, by making strategic choices for pair-wise grouping of numbers, and by appealing to arithmetic sequences and properties of such sequences.

Figure 3, continued.

Finally, we organized the set of mathematical actions to account for the actions arising in the Situations as well as reasonable mathematical actions that were not captured in the categories that were derived from the Situations. The final set of categories is displayed in Figure 4.

I. Mathematical noticing: *Recognize and choose from among known mathematical entities or settings based on known mathematical criteria such as:*

- A. Structure of mathematical systems
- B. Symbolic form
- C. Form of an argument
- D. Connections within and outside mathematics

II. Mathematical reasoning**:** *Reason about a mathematical entity in one or more than one way, including, but not limited to: from mathematical definitions, from given conditionals, from and toward abstractions, by continuity, by analogy, and by using structurally equivalent statements.*

- A. Justifying/proving
- B. Reasoning when conjecturing and generalizing
- C. Constraining and extending

III. Mathematical creating*. Create (Creating a mathematical entity or setting from known (to the one creating) structure, constraints, or properties)*

- A. Representing
- B. Defining
- C. Modifying/transforming/manipulating

IV. Integrating strands of mathematical activity. *Coordinate (Coordinate mathematical knowledge, student mathematical thinking, school curricula, and knowledge development); Reflect (self-reflect) (Reflect on mathematical aspects of one's practice or on one's own doing math); and Apply (Employ algorithms, definitions, and technology in mathematical settings and/or real world quantitative settings when applicable.)*

Figure 4. Mathematical Activity Perspective of the Framework for Mathematical Understanding for Secondary Teaching (Heid & Wilson, in press).

The final categories differed from existing frameworks in their mathematical nature. The mathematical actions we described derived from the mathematical decisions that teachers confront. Their work in mathematics classrooms would benefit from their ability to notice similar mathematical structures. Being comfortable enough with mathematical entities, properties, and structures to create and modify new representations would allow them the freedom to pursue their students thinking. They could productively use a flexible and robust repertoire of techniques for justifying their mathematical work.

The framework is intended to be a work in progress. It can serve as a research tool to study the mathematical understanding of secondary teachers. Researchers might investigate, for example, what collegiate mathematics courses contribute to the development of the capabilities suggested in each of the perspectives. They might also investigate how the aspects of secondary mathematics teachers' own mathematical understandings as described in the Framework influence the mathematics to which they expose their students.

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