Student interpretation and justification of "backward" definite integrals

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The definite integral is an important concept in calculus, with applications throughout mathematics and science. Studies of student understanding of definite integrals reveal several student difficulties, some related to determining the sign of an integral. Clinical interviews of 5 students gleaned their understanding of "backward" definite integrals, i.e., integrals for which the lower limit is greater than the upper limit and the differential is negative. Students initially invoked the Fundamental Theorem of Calculus to justify the negative sign. Some students eventually accessed the Riemann sum appropriately but could not determine how to obtain a negative quantity this way. We see the primary obstacle here as interpreting the differential as a width, and thus an unsigned quantity, rather than a difference between two values.

Key words: Definite integrals, Calculus, Differential

In this preliminary report, we examine the role of the differential in the "backward" definite integral, $\int_b^a f(x)dx$ where $a < b$. The definite integral is a fundamental concept in calculus, with applications throughout mathematics and science. Studies of student understanding of definite integrals reveal several difficulties (Bajracharya, Wemyss, & Thompson, 2012; Bezuidenhout & Oliver, 2000; Jones, 2013; Lobato, 2006; Sealey, 2006, 2014; Sealey & Oehrtman, 2005). The existing literature on definite integrals tends to support a specific approach to developing an understanding of the definite integral, specifically by recognizing it as the sum of infinitely small products, which are formed via Riemann sums (Jones, 2013; Meredith & Marrongelle, 2008; Sealey, 2008, 2014). Additionally, Sealey (2006) and Jones (2013) point out that recognizing the Riemann sum as a sum of products of the function value $f(x)$ and the increment on the *x*-axis (Δx) is necessary for students to understand the meaning of the *area under the curve*, which is, arguably, the most prominent metaphor/interpretation of the definite integral. On the other hand, reasoning about a definite integral as area under the curve may limit students' ability to apply the integral concept (Norman & Prichard, 1994; Sealey, 2006; Thompson & Silverman, 2008).

Another aspect of the definite integral that leads to student difficulties is the meaning of the differential itself. Students treat the differential as an indicator of the variable of integration rather than a fundamental element of the product in integration of both single- and multivariable functions (Hu & Rebello, 2013; Jones 2013). This could stem from a failure to understand the product layer of the integral (Sealey, 2014; von Korff & Rebello, 2012). Other recent work has shown students treating *dx* as a width rather than a difference or change, both for positive and negative integrals (Bajracharya et al., 2012; Hu & Rebello, 2013; Wemyss, Bajracharya, Thompson, & Wagner, 2011).

Interpreting the sign of the integral has been shown to be difficult for students. In particular, definite integrals that have a negative result are of particular difficulty geometrically. Students often do not treat the area as a negative quantity, effectively associating it with spatial area rather than the quantity represented by the product of $f(x)\ dx$. This is true for integrals for which $f(x)$ is negative, i.e., below the *x*-axis (Bezuidenhout & Oliver, 2000; Lobato, 2006), as well as those for which *dx* is negative, i.e., the direction of integration is in the negative direction (Bajracharya et al., 2012). The former type of negative integral is more common, but the latter also has relevance to applications in physical situations (e.g., finding thermodynamic work during the compression of a gas). Bajracharya et al. (2012) found that students could justify the sign of a negative integral represented graphically by overlaying a physical context on the graph.

The notion of *dx* as a signed quantity is somewhat controversial, depending on the way one defines the differential. The perspective here, which is consistent with applications in physics and other fields, is that *dx* is defined as an infinitesimal *change* in the quantity *x*, akin to the limit of the change in *x* for the products in a Riemann sum: $\Delta x = \frac{b-a}{n}$; $dx = \lim_{n\to\infty} \frac{b-a}{n}$. This is consistent with von Korff & Rebello (2012), who argue that infinitesimal quantities and infinitesimal products are important for an understanding of the meaning of definite integrals. Generally the sign of these quantities is not of interest, since $b>a$ in most cases. However, if $b<a$, then Δx, and thus *dx*, are negative. In Stewart's (2007) most recent text, he explains that the backward integral is negative because Δx is negative, but does not explicitly refer to *dx* as a signed quantity.

Given the prior work in this area, we wanted to explore the facets of students' concept image (Tall & Vinner, 1981) of the definite integral that apply to the sign of the integral. In particular, the role of the differential in a backward integral, $\int_b^a f(x)dx$, is crucial in interpreting the sign. We suspected that students would not recognize the fact that the differential would be negative for backward integrals. Thus the backward integral had the potential to illuminate students' understanding of the meaning of differentials, definite integrals, and to some extent, the Riemann sum, beyond what has been seen in the literature to date.

Methods

During clinical interviews, students were asked a series of questions about the relationship between forward and backward integrals. As this was a pilot study, we chose to interview five students at various levels: two second-semester freshmen (both double majors in math and physics and concurrently enrolled in a second-semester calculus course), one junior math major, one senior math major, and one first-semester Ph.D.-level graduate student in math/math education. Interviews were videotaped and transcribed. The interview subjects were volunteers who were either former students or teaching assistants of one of the authors. Interviewees received a $10 gift card at the conclusion of the interview. Prior to the interviews, we developed an interview protocol and agreed upon the order in which the questions would be asked of the students, starting with the open ended general expressions shown below and concluding with a physical example. In each case we gave the forward integral first, then asked about the backward integral of the same expression.

1. General expressions: $\int_a^b f(x)dx$ and $\int_b^a f(x)dx$
2. Specific expressions: $\int_1^3 2x\,dx$ and $\int_3^1 2x\,dx$
3. Physical scenario: Work required to stretch a spring, $\int_{x_1}^{x_2} F\,dx$, where $F = kx$

Data and Results

All five students were able to use the Fundamental Theorem of Calculus (FTC) to justify why $\int_a^b f(x)dx = -\int_b^a f(x)dx$. Specifically, they were able to state that $\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is the antiderivative of $f(x)$, and then that $\int_b^a f(x)dx = F(a) - F(b)$, which would have the opposite sign. Graphically, the students had much more difficulty. In the preliminary analysis, several student difficulties were observed; two of these are discussed in more detail here. We are still in the process of analyzing the data and determining plans for future data collection.

Student thinking about the differential

Most of the students were able to think about *dx* in at least two ways. Many of the students mentioned that the *dx* refers to the variable of integration, and most also were able to discuss the *dx* as the width of individual rectangles under a curve. Subsequent data analysis will note which concept image for *dx* was evoked in different circumstances, which concept image was evoked first, and if/when the students changed the way in which they thought about the *dx*. Of particular interest to us is whether or not the students can conceive of *dx* as a signed quantity, as either a negative width, or as a negative value obtained from $x_2 - x_1$. According to our preliminary analysis, none of the students thought about *dx* as a signed quantity on their own accord, but with prompting from the interviewers, some were able to do so.

Anna, a senior math major, had no trouble thinking about Δx as a negative width, but did not seem comfortable thinking about *dx* being positive or negative. Her explanation of why the backward integral was negative was because the width was negative, and explained, "You're going to have that negative width times a positive value, which is going to give you a negative number, so you're going to get the addition of a bunch of negative numbers." Much later in the interview, one of the interviewers asked Anna if it was possible for *dx* to be positive or negative, and Anna responded, "I've actually never thought of that. So I'm not sure. I mean I guess it could, but I just always viewed the *dx* as the indication of what term to integrate to. So I'm not actually sure, I guess."

Similar to Anna's response, Matt, a junior math major, eventually was able to think about Δx as a negative quantity and described *dx* as the limit as Δx approached zero. After many attempts from Matt, the interviewer asked him if *dx* could be negative. His response indicated that he was not confident in his answer, but responded, "That's probably the hidden spot that I couldn't figure out before. Yeah I would say that this *dx* would be negative (from a to b) and this one would be positive (from *b* to *a*) because it's approaching 0 so this (from a to b) would still stay positive like stay right north of 0. And this one (from *b* to *a*) would stay under, yeah I'm going to say this *dx* here (from *b* to *a*) is negative and this *dx* is a positive *dx* (from a to b), and I guess that's where it's hidden and that's what their difference is? I don't know."

Nick, a mathematics graduate student, focused his explanation as to why the backward integral was negative on direction. He said that the *dx* represents a change, and that change implies motion. He seemed to be thinking about the variable *x* representing time, and mentioned more than once that the backward integral would be like playing a movie in reverse. On another note, Nick spent a great deal of time during the interview talking about the two terms that made up the product in the definite integral, namely the 2*x* and the *dx* in $\int_3^1 2x\, dx$. He knew that when multiplying two quantities to obtain a negative result, exactly one of the terms multiplied must be negative. He debated if the *x* turned negative or the *dx* turned negative. He "voted" for the *dx* to be negative, but didn't seem confident of his answer. He said to be sure, he would have to go back to the definition of Δx in the textbook to see if he was right.

Using area under the curve and the Fundamental Theorem of Calculus

All of the students seemed comfortable discussing the integral as the area under the curve. While they were able to consider the total area as the sum of small rectangles (or trapezoids), their calculation of the total area ended up being an interesting part of our analysis.

Sara, a sophomore mathematics and physics double major, evaluated $\int_1^3 2x\,dx$ by finding the area of the large triangle (Fig. 1a) and subtracting the area of the small triangle (Fig. 1b) to obtain the desired area (Fig. 1c). She noticed that these calculations corresponded to the values she obtained when applying the FTC to the same problem: the area of the large triangle corresponded to $F(3)$, and the area of the small triangle to $F(1)$. Then, when computing $\int_3^1 2x\,dx$, she reversed the order of her subtraction, subtracting the area of the large triangle (Fig. 1a) from the area of the small triangle (Fig. 1b), and said, "But I'm not sure why that order is. I mean I know why for the integral [symbolically] because it's written that way, but if you were to solve this geometrically, I don't know why you would change the order of the subtraction."

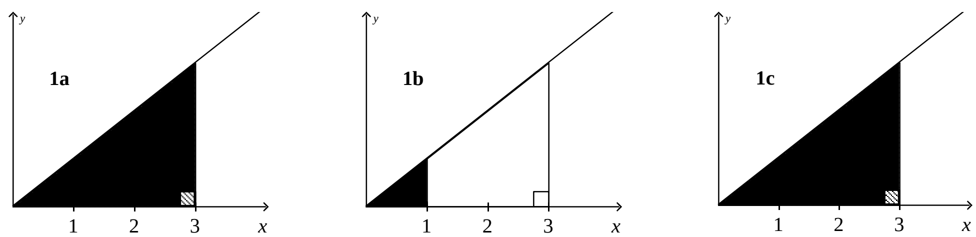

Figure 1: Sara's method of computing the area

Matt also was able to justify the relationship between the forward and backward integral symbolically using the FTC, but also struggled to justify the result graphically. When computing the area under the function $2x$ between $x = 1$ and $x = 3$, he recognized it as a trapezoid. Instead of using Sara's method of subtracting the smaller triangle from the larger triangle (Fig. 1), Matt added the area of the lower rectangle (Fig. 2a) to the area of the upper triangle (Fig. 2b) to obtain the total area (Fig. 2c).

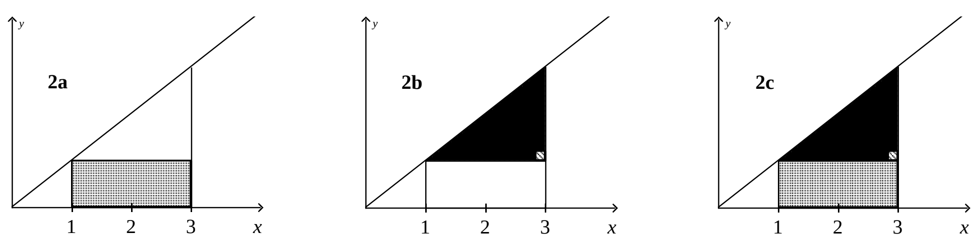

Figure 2: Matt's method of computing the area

Matt's solution is perfectly valid, but did not mimic the calculations from the FTC, as did Sara's method. Matt tried several different ways to graphically justify the negation of the backward integral but was never completely content with his justification. He noted that the backward integral represented the same area as the forward integral, but the backward integral would have to be negative since the limits were reversed "because I already know that, like as a fact, that it's a negative if you want to flip the bounds." He did state that he believed there *should* be a graphical justification, but he did not know what one would be.

We do not mean to imply that Sara's solution was in some way better than Matt's, but simply note the connection to the FTC in Sara's solution. In fact, both Sara and Matt used solutions that sidestep the need for thinking about the Riemann sum and the *dx* specifically. Near the end of Sara's interview, we pushed her to consider each rectangle under the curve, which she had described at the beginning of her interview. Sara was comfortable with $f(x)$ being negative or positive, depending on if it was above or below the *x*-axis, but said, "Well no, I don't think *dx* would ever be negative because it's just a distance, it's not like an actual value."

Discussion

Students recognized the negative value of the backward integral based on the FTC/antiderivative difference formula, but when asked for a geometric interpretation, most said they hadn't thought about it before and had difficulty making a reasonable interpretation on their own. Most students' graphical explanation of why the backward integral yields a negative result seemed to be invoking the direction of the integration, treating the area as a macroscopic negative quantity, but failed to recognize the role of the differential in generating that sign. We know from the literature and our own prior research (Bajracharya et al., 2012; Sealey, 2006; Thompson & Silverman, 2008) that students often lack an understanding of *why* or *how* area under a curve is a representation of a definite integral. Our subjects, who we acknowledge may be more advanced than the average calculus student, did not seem to have this difficulty and were able to describe the definite integral as the sum of the areas of very small rectangles, and adequately described the product layer that makes up these small rectangles. They could all explain that $f(x)$ represented the height of the rectangles and that Δx (and sometimes *dx*) represented the width of the rectangle.

However, thinking about the backward integral adds another level of difficulty to describing the definite integral in terms of area. The students did not always recognize that Δx and *dx* could be negative values. Instead of thinking about Δx as a difference, (e.g. as $(x_{i+1} - x_i)$ or as $\frac{b-a}{n}$), they initially thought of Δx as the width of a rectangle, and usually assumed it was always a positive value.

We certainly do not mean to imply that Δx and *dx* should never be thought of as a width. In fact, research by Hu and Rebello (2007) suggested that *dx*-as-width is an important perspective for problem solving in physics. Instead, we emphasize the necessity for being able to think about *dx* as positive or negative widths *and* the change between two quantities. With moderate prompting, most of our research subjects were able to do this, and our future research will examine what type of instruction or intervention enables students to make this connection.

Discussion Questions

1. We have some examples in physics where one might consider the backward integral (stretching/releasing a spring). Are there other examples *in mathematics* where it makes sense to consider $\int_b^a f(x)dx$?
2. Where do you think this difficulty might be best addressed? Calculus 1? Calculus 1? Real analysis? Physics?
3. Student functional understanding of the differential seems to be the underlying cause of several difficulties with students (in our work as well as other studies in the literature). Do you have recommendations for why and/or how this can be improved?

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