Mary, Mary, is not quite so contrary: Unless she's wearing Hilbert's shoes

Researchers (Leron, 1985; Harel & Sowder, 1998) have argued that students' lack a preference for indirect proofs and have argued that the lack of preference is due to a preference for constructive arguments. Recent empirical research (author, 2015), however, which employed a comparative selection task involving a direct proof and an indirect proof of the contraposition form, found no evidence of a lack of preference for indirect proof. Recognizing that indirect proofs of the contradiction form may differ from those that employ the contraposition, this study documents students' proof preferences and selection rationales when engaging in a comparative selection task involving a direct proof and an indirect proof of the contradiction form.

Key words: Indirect proof, Proof preferences, Proof by contradiction

It has been argued by many that indirect proofs, that is, proof by contraposition and proof by contradiction, are particularly difficult for students (Tall, 1979; Robert & Schwarzenberger, 1991) and that students' difficulties are related to a lack of preference for these forms of proof (Leron, 1985; Harel & Sowder, 1998). Several reasons for students' difficulties and lack of preference have been proposed. Tall (1979) conducted an empirical study of 37 students' levels of confusion in relation to proofs by contradiction of the irrationality of the $\sqrt{2}$ using an instrument that included the standard proof and two alternative proofs. He found that students experienced significantly lower levels of confusion with one of the alternative forms; namely, that which employed generic structures (i.e., proof structures that were not specific to the numbers used). Tall argued that use of generic proofs will aid students' understanding of indirect proofs. In a reflective account of multiple teaching experiments, Leron (1985) noted that not only are students perplexed by proofs by contradiction but that such proofs stand in contrast to much of students' mathematical activity, for they call on students to not only build up a "false world" but to destroy this world. Hence, according to Leron, students' difficulties are related to the coupling of non-constructive reasoning and a detachment from one's "real" mathematical world. Using the standard proof of the infinitude of primes, Leron reported that constructive approaches, which explore and analyze mathematical objects in their own right prior to their use as tools for obtaining contradictions, may enhance students' understanding of proofs by contradiction. Harel and Sowder (1998) have also argued that students are not convinced by proof by contradiction and lack a preference for this form of proof. Drawing of data from 6 teaching experiments they argue that students' dislike of indirect proofs represents a particular manifestation of the constructive proof scheme: a scheme in which "students' doubts are removed by actual construction of objects – as opposed to mere justification of the existence of objects" (p. 272). Lastly, Antonini and Mariotti (2008) studied students' views and production of indirect proofs. Drawing on the theory of *Cognitive Unity* and a specific characterization of *Mathematical Theorems* (Mariotti, Bartolini Bussi, Boero, Ferri & Garuti 1997), this research has sought to explore: (a) linkages between students' informal, indirect geometric arguments in technological environments and their production of proofs by contradiction; and, (b) the nature of students' difficulties with indirect proofs. Specifically, within their work a distinction is made between mathematical theories (e.g., Euclidean geometry; Riemannian geometry; Number theory) and metatheories (e.g., Standard logic, Constructive logic). Drawing on interviews with university students, Antonini and Mariotti demonstrated that students' difficulties with indirect proof may be tied to students' lack of acceptance of metatheorical properties (e.g., $P \rightarrow Q \equiv \sim Q \rightarrow \sim P$). For instance, when presented with a proof by contraposition of the statement, "If n^2 is even then n is even," students readily accepted the contrapositive proof as a proof of the statement, "If n is odd then n^2 is odd" but struggled to accept the proof as a proof of the original statement. Speaking to this issue, a student remarked, "… The problem is that in this way we proved that n is odd implies n^2 is odd, and I accept this; but I do not feel satisfied with the other one" (p. 407). Antonini and Mariotti's work is novel, for their work is the only research that proposes students' lack of acceptance of indirect proofs may be due to metatheoretical issues.

Four aspects of research on students' difficulties and lack of preference for indirect proof are noteworthy. First, research on students' difficulties with indirect proof is unique in that it is the only area of research within the broad spectrum of research on students' difficulties with proof in which researchers have linked students' difficulties to a lack of preference for that form of proof. Second, while researchers (Tall, 1979, Healy & Hoyles 2000, Knuth, 2002) have routinely engaged students in comparative selection tasks to determine which form of proof students' find most convincing, researchers have not examined students' preferences (or lack of preference) for indirect proofs using comparative selection tasks involving a direct and an indirect proof. Indeed, there is a scarcity of empirical evidence to support current claims regarding students' lack of preference. Third, while Antonini and Mariotti (2008) have provided evidence of students' lack of acceptance of metatheoretical statements there is the question of whether it is a lack of acceptance or a lack of recognition of these statements that is prevalent and at the root of students' difficulties. Fourth, current accounts of students' dislike of indirect proofs and preference for constructive and generic proofs have ignored the fact that these reactions may be the result of the mathematics community's practices related to introducing novices to indirect proofs and the discourse that occurs around such proofs. For instance, in *How to Solve It*, a famous problem solving text by Polya (1957), the section on *reductio ad absurdum* and indirect proof[1] concludes with a section titled "Objections," in which Polya states:

> We should be familiar both with 'reductio ad absurdum' and with indirect proof. When, however, we have succeeded in deriving a result by either of these methods, we should not fail to look back at the solution and ask: Can you derive the result differently (p. 169).

Arguably, Polya's remarks do not provide the reader with a strong endorsement of either method. Moreover, such sentiments are not difficult to obtain as illustrated by the textbook excerpts shown in Figure 1.

[Concluding remarks, section on proof by contradiction] Many mathematicians feel that if a result can be verified by a direct proof, then this is the proof technique that should be used, as it is normally easier to understand. Text: *Mathematical Proofs: A Transition to Advanced Mathematics* (Chartrand et al., pg. 132)
A proof by contradiction is often easier, since more is assumed true; you are able to assume both the hypothesis and the negation of the conclusion. On the other hand, a proof by contradiction is likely to be less elegant than a proof by contrapositive. In any case, for elegance and clarity, it is better to choose a direct proof over an indirect proof whenever possible. Text: *Introduction to Advanced Mathematics* (Barnier & Feldman, 2000, p. 43).
There are times when it is not easy to see how to prove a mathematical statement, say ψ. When this happens one should try the strategy called proof by contradiction. This strategy is perhaps the strangest method of proof. Text: *A Logical Introduction to Proof* (Cunningham, 2012, p. 93).

Figure 1. Proof by Contradiction Text Excerpts

[1] Polya refers to proof by contraposition as indirect proof and proof by contradiction by its Latin name, *reductio ad absurdum*.

[2] Hardy referred to reductio ad absurdum (proof by contradiction) as a mathematician's "finest

These excerpts are not meant as backing for the claim that the mathematics community as a whole has exhibited a lack of preference. Indeed, the writings of Hardy (1940/2005), Euclid, Archimedes, and many contemporary mathematicians, as well as the famous proofs by contradiction of Hilbert (*cf*. Hilbert, 1890), stand in contrast to the remarks shown above.[2] Instead, the excerpts illustrate how a lack of preference might be due to various enculturative acts rather than an attribute of students. Yet, much of the research on indirect proof has ignored students' rationales for either preferring or exhibiting a lack of preference for such proofs. To be certain, there is a need for research that not only documents students' comparative preferences but also students' selection rationales; that is, their reasons for choosing a particular proof form.

In (author, 2015), a study was reported in which 53 mathematics majors were surveyed using a comparative selection task (see Figure 2) involving a direct and a (contraposition-form) indirect proof of the following theorem: *Suppose a set A has the property, for any subset B, A* $\subseteq$ *B, then A* = $\varnothing$*.* The proofs were presented side-by-side and students were asked, "Which proof, in your opinion, is the most convincing? In other words, which proof better persuades you of the truth of the theorem" and "Please explain your selection." The two proofs in the selection task were designed so as to control for various proof features; namely, the proofs were similar in length, and designed with the intent to be equal in their level of familiarity and complexity. For instance, complexity was equated by the prevalence of the proofs' content in the same textbook chapters in multiple texts. These controls were employed because pilot work had shown that when either complexity or familiarity were not equated, each were individually predictive of students' selections regardless of the proof type (i.e., direct or indirect).

Theorem 3: Suppose a set A has the property, for any subset B, A $\subseteq$ B. Then, A = $\varnothing$.	
Proof A:	**Proof B:**
Suppose A $\neq \varnothing$. Then there exists an a, such that $a \in$ A. Hence, A$\not\subseteq \varnothing$. Thus, there exists a subset B for which A$\not\subseteq$ B.	Assume A has the stated property. Recall, that $\varnothing$ is a subset of every set. Thus, $\varnothing \subseteq$ A. By the given property, A $\subseteq \varnothing$. Thus, A = $\varnothing$.

Figure 2. Comparative Selection Task

Surprisingly, the survey results indicated that the *indirect:direct* selection ratio for the Theorem 3 proof comparative selection task was 27:26. Thus, no evidence of a lack of preference was found. This finding lies in contrast to the findings of prior research and raises several questions: (1) to what extent is a lack of preference prevalent; and, (2) if prevalent, what are the characteristics of contexts in which a lack of preference is manifested? Furthermore, analyses of the students' selection rationales demonstrated that students' primary rationales were *certainty* and *complexity*. *Certainty* refers to the degree to which a student is certain of his/her understanding of the given proof and *complexity* refers to students' identification of one proof as being more complex than the other. What is of particular interest is that students' rationales did not identify a "more complex" proof nor were students more certain of one proof than the other. Instead students' responses demonstrated that complexity and certainty were subjective; that is,

[2] Hardy referred to reductio ad absurdum (proof by contradiction) as a mathematician's "finest weapon."

dependent on the individual and his or her understanding of the content employed. Drawing on Balacheff's cK¢ theory, (author) argued that preferences are mediated by students' conceptions.

While providing grounds for questioning the extent to which a lack of preference is prevalent among undergraduate students, reasons to continue investigating students' preferences remain. To begin, proof by contradiction and proof by contraposition differ at the metatheoretical level, with contraposition proofs requiring a direct proof of the contrapositive statement and use of the logical equivalence, $(P \rightarrow Q) \equiv (\sim Q \rightarrow \sim P)$, while proofs by contradiction require learners to not only negate a conditional statement (which is arguably more difficult than negating a premise and a conclusion separately) but also to produce an unspecified contradiction and to correctly interpret the ramifications of that contradiction (e.g, as the negation of a negated statement rather than as an error). In the previous study, the Theorem 3 selection task engaged students in a *direct:indirect* proof selection involving a *direct proof:proof by contraposition* comparison. Hence, there is reason to question if the lack of definitive preference, as evidenced by the students' selection ratio, is predictive of students' preferences in comparisons involving a proof by contradiction; especially, given the differences cited above. With this said, there are cultures in which the two forms of proof (contraposition and contradiction) are not distinguished at a nominal level, *e.g.*, in Italian (*cf.* Antonini & Mariotti, 2008). Moreover, pilot data showed students' may categorize a proof by contraposition as a proof by contradiction. Consequently, it may be that students' lack of definitive preference when engaging in *direct proof:proofs by contraposition* comparisons is predictive of students' preferences during *direct proof:proof by contradiction* comparisons. Certainly, more research is needed. The aim of this study is to address this need by pursuing the following research questions:

1. Do undergraduate mathematics students exhibit a lack of preference for indirect proof, when engaging in comparative tasks involving both a direct proof and proof by contradiction?
2. Which rationales do students provide for their selection of the most convincing proof, when engaging in comparative tasks involving both a direct proof and proof by contradiction?

The Study

The research reported in this paper is part of a larger research program generally focused on undergraduate mathematics students': (a) development of hypothetico-deductive reasoning (Piaget, 1968/1964); and (b) emerging *conceptions* of indirect proof, where conception is used in the sense of Balacheff's cK¢ model (2010; 2013). To investigate students' preferences, as these relate to selecting the most convincing proof, 85 mathematics students were recruited and given a paper survey containing Theorem 3 and two proofs of the statement, which were a direct and an indirect proof of the contradiction form (see Figure 3). The form was similar to that used in the previous study, with two exceptions; namely, the indirect proof form and a slight adjustment to the wording of the direct proof so as to produce proofs with equated lengths, (as determined by word counts of 40 and 41 words). As was the case in the prior study, complexity and familiarity were viewed as equated due to the content occurring in the same chapter in multiple introduction to proof texts.

The surveys were administered in either an abstract algebra or analysis course. Students completed the surveys under the supervision of the researcher and returned the surveys directly to the researcher. Proof order was randomized to avoid a priming effect. Analyses of the data involved the determination of selection ratios and the coding of students' rationales using a constant comparative methodology (Creswell, 1994). Multiple codes were employed when multiple rationales were provided by the students.

Theorem 3: Suppose a set A has the property, for any subset B, $A \subseteq B$. Then, $A = \varnothing$.	
Proof A	Proof B
Assume A has the stated property. Recall, that $\varnothing$ is a subset of every set. Thus, $\varnothing \subseteq A$. By the given property, since $\varnothing$ is a subset of A, $A \subseteq \varnothing$. It follows from that, $A = \varnothing$.	Assume A has the stated property and $A \neq \varnothing$. If $A \neq \varnothing$ then $A \nsubseteq \varnothing$. By the given property, $A \subseteq \varnothing$. Since, it cannot happen that $A \nsubseteq \varnothing$. and $A \subseteq \varnothing$, it follows that, $A = \varnothing$.

Figure 3. Theorem 3 Contradiction-Form Comparative Selection Task

Results

Data from the Theorem 3 comparative selection task indicate that students found the direct proof more convincing than the proof by contradiction with a *direct:indirect* selection ratio of 56:29. Thus, the students' preferences differ considerably from those observed in the *direct:indirect* comparative tasks involving a proof by contraposition (*direct:indirect* selection ratio of 26:27). Analyses of students' selection rationales also indicate differences in students' comparative assessments. Specifically, while the vast majority of students' rationales focused on *certainty* and *complexity* when engaging in the contraposition comparison, the selection rationales for the contradiction comparison were more varied. Indeed, six rationales were present in students' written remarks, which are reported with students' selection ratios in Table 1.

Selection Rationale	Selection Ratio (Contra-d: Direct)	*n*	Percent of Students
Simplicity / Ease	7:21	28	32.9%
Error (in Alternative)	4:14	18	21.2%
Directness / Straightforward	4:23	20	31.8%
Matched My Thinking	9:8	17	20.0%
Familiarity	1:12	15	15.3%
Stronger Argument	6:4	12	11.8%

Table 1. Students' Selection Rationales

Due to space limitations, examples of students' rationales will be restricted to: *simplicity*, *error*, and *matched my thinking*. Though *directness* was common, it is not included as it is self-evident in meaning. Below (Figure 4) are two examples of students' rationales coded as *simplicity*.

[Example 1.] The second is a proof by contradiction. I tend to find these proofs easier to follow. Proof A is not hard to follow as well but I think, in general, proof by contradiction is easier. (Selection: Indirect)
[Example 2.] Proof A is simpler. Proof B forces the reader to think about it more deeply. (Selection: Direct)

Figure 4. Simplicity Rationales

As can be seen by these remarks, students viewed both proofs as simple. However, as indicated by the *direct:indirect* selection ratio of 21:7, the simplicity rationale was more prevalent among students who selected the direct proof. The code *error* was used to denote student rationales that indicated a proof contained an error or that there was a statement that the student was uncertain about. Below (Figure 5.) are two examples of student rationales coded as *error*.

[Example 3.] Proof A stated the property that we need to prove and we cannot do that. (Selection: Direct)
[Example 4.] Proof A states that condition as an assumption which immediately made me question the validity of the proof. Proof B follows a standard version of a proof and make more sense than A. (Selection: Direct)

Figure 5.

As indicated by these students' rationales, there was a tendency among some students to view the contradiction argument as flawed. Indeed, the data indicate that 14 students (16.5%) selected the direct proof and provided this rationale. This finding suggests that rather than lacking a preference for indirect proof, students may have difficulty comprehending and/or validating indirect proofs of the contradiction-form.

The code *matched my thinking* was used for rationales that focused on students' statements of an alignment between their own approaches to proving and that taken in the selected proof. Four examples, which illustrate students' remarks, are provided in Figure 6.

[Example 5.] While thinking about how I would prove this theorem, Proof B seemed to match what I would have said. (Selection: Direct)
[Example 6.] Personally, I like working with direct proofs rather than contradictions. In Proof B the logic makes sense. (Selection: Direct)
[Example 7.] It was contradiction and I like to use contradiction to solve proofs. (Selection: Contradiction)
[Example 8.] When something seems obvious or believe it's true, it's easier for me to assume not and follow that way. (Selection: Contradiction)

Figure 6.

These responses, accompanied by a *direct:indirect* selection ratio of 8:9, suggest that the 20% of students who attended to their own approaches to the theorem (i.e, their habits of reasoning) did not demonstrate a preference for the direct proof by rather lacked a dominant preference. While the sample size for this rationale (and the others) is small, one must question if a preference would be evident in a larger data set. Nevertheless, it is particularly interesting that those who attended to their own approaches did not demonstrate a direct proof preference. Lastly, a note regarding the familiarity code is warranted. This code was used to indicate rationales focused on students' who cited familiarity with containment arguments (*e.g.*, $A \subseteq B$, $B \subseteq A$, thus $A = B$) and their recognition that the direct proof employed a known proof technique.

Discussion

Findings from the survey suggest, as indicated by prior research (Leron, 1985; Harel & Sowder, 1998), that students may lack a preference for indirect proofs of the contradiction-form. Moreover, when these findings are considered in relation to the contraposition-form results, were no preference is evident, it appears that the two forms of proof are not the same in the eyes of undergraduate mathematics students. With this said, there are reasons that claims related to a lack of preference should be stated with caution. First, while the *indirect contraposition-form:direct proof* comparative selection task did not elicit the *error* rationale, this rationale was proposed by 16.5% of students in relation to the contradiction proof during the *indirect contradiction-form:direct proof* comparative selection task. Thus, it may be the case that students are more prone to comprehension difficulties with contradiction proofs rather than lack a preference for this form of proof. Second, while Tall (1979), Knuth, (2002), and Healy & Hoyles (2000) all reported results in which students' proof selections were impacted by familiarity, it was only the latter experiment, where a *contradiction-form:direct proof* comparison selection task was used, that students employed a *familiarity* rationale and stated that the direct proof employed a known technique. Familiarity is interesting in that while cognitive psychologist have argued that familiarity can create an immediate "feeling of rightness" it may also be the case that familiar proof forms are selected because students know those proof forms are accepted by the mathematics community or believe that the alternative is less favorable – a belief that could arise from reading texts like those in Figure 1. Thus, it is unclear if students' task-specific inclusion of the *familiarity* rationale is due to students' seeking "feelings of rightness," a type of deference to the community's argumentation norms, or something else. Certainly, more research is needed.

Furthermore, since familiarity strongly influences preferences and claims of preference have been predicated on assumptions of comprehension, it is worth examining those students who neither viewed the contradiction-form proof as flawed (*error* rationale) nor cited *familiarity*. Among the 85 students surveyed, 12 reported familiarity with containment arguments as their primary rationale and 1 reported familiarity with the contradiction proof. Additionally, while 14 reported an error in the contradiction argument, only 4 students viewed the direct proof as flawed. Removing these two categories of students from the population reduces the *direct:indirect* selection ratio[3] of 56:29 to a selection ratio of 31:24. Thus, the proportion of students selecting the direct proof (0.56) is not statistically significantly different from a 0.5 proportion ($z = 0.067$, $p < 0.05$). To be certain, among those who did not demonstrate a lack of comprehension and who did not defer to a "known technique," there is little evidence of a preference for the direct proof. Thus, while far from providing a definitive conclusion, this research raises multiple questions regarding students' preferences for or against proof by contradiction – perhaps with the exception of those who, like Hilbert, developed contradiction as a habit of reasoning.

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[3] This ratio is adjusted for the multiple codes used when students provided multiple rationales. In other words, there is no double counting of students in the adjusted preference ratio.

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