When nothing leads to everything: Novices and experts working at the level of a logical theory

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Building on Antonini and Mariotti's (2008) theorization of mathematical theorem and research on students' meta-theoretical difficulties with indirect proof, this study examines mathematics majors' and mathematicians': (1) responses and approaches to the validation tasks related to the assertion $S^* \rightarrow S$, when given a primary statement, S, of the form $\forall n, P(n) \Rightarrow Q(n))$ and a secondary statement, S, of the form used in proofs by contradiction; namely, $\nexists n, P(n) \wedge \sim Q(n))$; and, (2) selection of a statement to prove given the choices S* and S. Findings indicate that novice proof writers' responses differ from advanced students' and mathematicians' both in their approaches and selections, with novices tending to become entangled in natural language antonyms and engage in the chunking of, rather than parsing of, quantified statements.*

Key words: Indirect proof, Proof by contradiction, Meta-theoretical Difficulties

How does one know that a mathematical theorem is true? Mariotti (2006) has proposed that to *know* truth in a mathematical sense requires not only a mathematical theory but also a logical theory:

> In their practice, mathematicians prove what they call 'true' statements, but 'truth' is always meant in relation to a specific theory. From a theoretical perspective, the truth of a valid statement is drawn from accepting both the hypothetical truth of the stated axioms and the fact that the stated rules of inference 'transform truth into truth' (p. 184).

These remarks align with definitions of proof, which focus on the elements of proofs and their use of logic; such as that proposed by Akin (2010), "Proofs are sequences of statements which can, in theory, be reduced to: (1) axioms, definitions, and previously proved results; or (2) statements obtained from earlier statements by the (formal – logical) rules of inference. To be certain, this definition points to the fact that without a mathematical theory from which to draw axioms, definitions, and previously proved results and without a logical theory to guide inferences, one cannot produce mathematical proofs. Indeed, we would obtain quite distinct results working in Riemannian rather than Euclidean geometry; especially if we were to use intuitionistic logic, such as that used by Brouwer, rather than the standard logic of mathematics. Working to clarify the systems that make 'truth' possible in mathematics Mariotti, Bartolini Bussi, Boero, Ferri & Garuti (1997) have argued that what characterizes a *mathematical theorem* is the triplet (statement, proof, reference theory), where reference theory is used to describe "a system of shared principles and deduction rules" (p.8).

Building on this characterization of mathematical theorem, Antonini and Mariotti (2008) examined a form of proof researchers (Robert, & Schwarzenberger, 1991) describe as highly problematic for students; namely, indirect proof. Drawing on data from interviews with tertiary students, Antonini and Mariotti demonstrated that while students may gain conviction from indirect proofs, this conviction is tied to the specific statements proved, as opposed to their logically equivalent statements. For instance, given a proof of $\sim Q \rightarrow \sim P$, students may gain conviction but fail to do so in relation to the statement $P \rightarrow Q$. Drawing on these findings, Antonini and Mariotti argue that when examined through the lens of that which characterizes mathematical theorems, indirect proofs are unique for they call on learners to employ theorems not only within the mathematical theory but also within the logical theory. Consequently, Antonini and Mariotti proposed a refinement of the (statement, proof, reference theory) triplet for indirect proofs, arguing that indirect proofs involve "the pairing of the sub-theorem (S*, C, T) and the meta-theorem (MS, MP, MT)" (p. 405). Meta-statement (MS) refers to statements such as "$S \rightarrow S^*$" where S refers to a primary statement (e.g., $P \rightarrow Q$) and S* refers to a secondary statement, such as the contrapositive of S (i.e., $\sim Q \rightarrow \sim P$). Meta-proof (MP) refers to the proof of $S \rightarrow S^*$ within the meta-theory (MT), i.e., the logical theory. Drawing on this model, Antonini and Mariotti argue students' difficulties with indirect proofs are metatheoretical; that is, due to a lack of acceptance of the meta-theorems employed.

Antonini and Mariotti's (2008) account of students' difficulties with indirect proof is unique among accounts of students' difficulties. Indeed, while researchers have argued that part of students' difficulties with indirect proof arise from difficulties negating statements (Wu Yu, Lin & Lee 2003; Antonini 2001, 2003; Thompson 1996), it is also the case that few have attended to the role of logic in such proofs. In Tall's (1979) study of students' levels of confusion related to indirect proofs of the irrationality of $\sqrt{2}$, students' confusion was attributed to the lack of generic structures in standard contradiction proofs rather than to difficulties at the level of the logical theory. One reason for this may be that the particular statement Tall (1979) examine was not a compound statement. Hence, the difficulties associated with negating conditional statements were not relevant. Drawing on reflective accounts of multiple teaching experiments, Leron (1985) explored students' responses to standard proofs of the infinitude of primes – that is to a mathematical theorem which did not take the form of a compound statement – and found that students' experience difficulties with the destructive, as opposed to constructive, nature of the indirect proof, as well as the "negative stretches" that accompany working in a "false world." Hence, working within the logical theory was not a primary source of difficulties. Through analyses of data from multiple teaching experiments, Harel and Sowder (1998) examined students' reactions to contradiction proofs and found that students lack a preference for this particular form of proof. They argued, like Leron (1985), that students' prefer constructive approaches and that such preferences are indicative of a constructive proof scheme. Beyond the rationales of Tall (1979), Leron (1985), and Harel and Sowder (1998), are Reid and Dobbin's (1998) emotioning rationale and Thompson's (1996) argument that indirect proof instruction tends to lack connections to students' informal reasoning. Indeed, multiple rationales have been proposed, though few take into account the specific meta-theorical nature of indirect proofs. This has been the case even though Goetting (1995) noted in relation to proof by contraposition, students were "wary of the validity of the 'backwards' arguments" (p. 124) and Leron (1985) noted in the case of proofs by contradiction, "we must be satisfied that the contradiction has indeed established the truth of the theorem (having falsified its negation), but psychologically, many questions remain" (p. 323) – remarks suggestive of attention to metatheoretical issues.

Arguably, Antonini and Mariotti's (2008) metatheoretical rationale represents a potentially critical advance in research on indirect proof in that it offers a route by which to explore aspects of indirect proof that are both unique and essential to that form of proof. With this said, little is known about tertiary students' metatheoretical reasoning. That this is the case may be due to the fact that there are a plethora of studies that focus on the development of proof through the refinement of students' informal arguments and that argues against the direct transition to formal proof and, consequently, do not advocate approaches involving training in logic (Jahnke, 2010; Maher & Martino, 1996). Moreover, many "Introduction to Proof" texts include instruction on logic but focus on building students' understandings through natural language activities rather than through instruction on logic as a theory (*cf.* Chartrand et al., 2013). Thus, one could conclude that there is little interest in metatheoretical issues for they run counter to current perspectives on productive approaches to proof and pedagogical practices in commonly used texts. On the other hand, the lack of research on metatheoretical reasoning is surprising since, as evident in Akin's definition, our basic definitions of proof rely on the existence of a logical theory. Furthermore, there is a profusion of research from cognitive psychology demonstrating that humans' ways of reasoning do not fully align with the forms of reasoning used in standard logic (*cf.* Oaksford & Chater, 2010). Specifically, general tendencies for interpreting conditional statements ($P \rightarrow Q$), do not align with those used in mathematics, with the exception of direct reasoning processes; e.g., accepting $P \rightarrow Q$ when P and Q are true. Thus, there are grounds for questioning pedagogical approaches premised on the idea that the logical theorems employed in mathematics will be readily employed by students. To be certain, there is reason to argue that further research is needed on students' approaches to and extent of success with metatheoretical work, especially in relation to those forms of proof for which such work is essential; namely, indirect proofs.

The Study

The purpose of this study is to address the need for research on students' approaches to and extent of success with metatheoretical work, by exploring three research questions:

1. To what extent are mathematics majors and mathematicians successful, when answering questions regarding the validity of $S^* \rightarrow S$; that is, when asked if a secondary statement S^* is sufficient to prove a primary statement S?
2. What are the similarities and differences observed among these populations when approaching questions regarding the validity of $S^* \rightarrow S$?
3. Which formulation, the statement or the secondary statement, do mathematics majors and mathematicians prefer, when asked to select a statement to prove?

These questions are of interest for they provide various avenues with which to explore the issue of students' potential difficulties working at the level of a logical theory. Indeed, if experts (mathematicians) are able to successfully verify metatheoretical statements, while novices struggle, then the findings will provide further evidence of Antonini and Mariotti's (2008) claim; whereas, if neither experience difficulties then an alternative to the metatheoretical hypothesis may be needed. Furthermore, if experts and novices avoid selection of secondary statements, then such data will provide further evidence of the preference hypotheses generated by Leron (1985) and Harel and Sowder (1998). In contrast, if no preference is evident then there will be cause to question the preference hypothesis.

Methods

To investigate mathematics students' and mathematicians' metatheoretical reasoning, three stages of data collection occurred. In the first stage, electronic surveys were sent to undergraduate mathematics majors who were within one year of completion of an introduction to proof course. The students were provided with Theorem 5 and Statement A (see Figure 1.) and asked to indicate if the following statement was true or false, "You can prove Theorem 5 by proving Statement A." Following this prompt, students were queried "If you were asked to prove Theorem 5 which would you pursue first?" and given the choices Theorem 5 and Statement A.

Theorem 5. For all positive integers n, if $n\text{mod}(3) \equiv 2$ then n is not a perfect square.
Statement A. There exists no positive integer n such that $n\text{mod}(3) \equiv 2$ and n is a perfect square.

Figure 1. Theorem 5 Task

In stage 2, clinical interviews were conducted with 21 mathematics majors, who met the criteria described above. In stage 3, clinical interviews were conducted with 6 mathematicians. In all of the clinical interviews, the manner in which the Theorem 5 task was posed matched that of the electronic survey. Interview responses were analyzed to gather categorical data for the validation and selection questions, as well as data regarding participants' approaches to the validation question. Using a constant-comparative methodology, descriptive codes were generated and used to further characterize participants' video-recorded responses.

Theorem 5 Task Design

The Theorem 5 validation task was designed with the assumption that at the conclusion of a proof by contradiction the prover would need to recognize that having shown Statement A, he or she can conclude that Theorem 5 was proven. This assumption is predicated on the following sequence of proving actions. First, to prove Theorem 5 using a proof by contradiction, one begins by assuming the negation of a statement of the form $\forall n \in \mathbf{Z}^+$, $P(n) \Rightarrow Q(n)$. Thus, by assuming $\sim(\forall n \in \mathbf{Z}^+, P(n) \Rightarrow Q(n))$, which is logically equivalent to $\exists n \in \mathbf{Z}^+$, $P(n) \wedge \sim Q(n)$. Second, the prover must arrive at a contradiction to an axiom, definition, previously proved theorem or an existing assumption (*i.e.*, something within the mathematical theory) and conclude from this contradiction that the statement $\exists n \in \mathbf{N}$, $P(n) \wedge \sim Q(n)$ is false; *i.e.*, $\sim(\exists n \in \mathbf{N}, P(n) \wedge \sim Q(n))$. While formally, one might say "It is not the case that there exists a positive integer such that …", informally one might argue, "no such n exists" or the more common, though grammatically more awkward, "there exists no positive integer n such that …". Lastly, one must recognize that the proven statement is sufficient to prove the original statement – in Antonini and Mariotti's terms, that S* proves S. Hence, the Theorem 5 validation task was designed with the last phase of this sequence in mind; that is, to explore mathematics majors' and mathematicians' perceptions of validity related to a standard phrasing of the result of a proof by contradiction. Furthermore, as was the case in Antonini and Mariotti's work, the task focused on the inferences to be drawn rather than the, perhaps technically more appropriate, determinations of equivalence.

Findings

In regard to the electronic survey, 35 mathematics majors who were advanced in terms of course work; that is, who had completed not only an introduction to proof course but also real analysis and abstract algebra courses, responded to the survey. The categorical responses are shown in Table 1 and indicate that: (a) the majority of advanced students were successful at recognizing the validity of the metatheoretical statement $S^* \rightarrow S$; and, (b) demonstrated only a slight preference for the primary (Theorem 5) over the secondary statement (Statement A), which was not significantly different from a near even split ($\chi^2 = 0.458$; $p = 0.499$, χ^2 *Good of Fit test*).

Prompt 1: *You can prove Theorem 5 by proving Statement A.*		
Response	**N**	**%**
True	29	83
False	6	17

Prompt 2: *If you were asked to prove Theorem 5, which would you pursue first?*		
Responses	**N**	**%**
Theorem 5	20	57
Statement A	15	45

Table 1. Advanced Mathematics Students' Survey Responses

While the surveys were predominantly taken by advanced students, the majority of interview volunteers were novice proof writers; *i.e.*, they had recently completed an introduction to proof course, had not completed both real analysis and abstract algebra courses but rather were enrolled in at most one of these courses at the time of the interviews. Thus, the interviews were conducted with a different population in terms of prior coursework. Findings from the interviews indicate that the novices' validation responses fell into four categories: *don't know*, *no*, *yes-no-yes*, and *yes*. For clarification, *don't know* was used for students who after deliberating indicated they were unable to determine if the statement "You can prove Theorem 5 by proving Statement A" was true or false. *Yes-no-yes* refers to responses in which the student initially articulated an intuitive response, sought to validate their intuition, decided "*no*" and then through further analyses decided (often with uncertainty) that their response was, "Yes, it's true." As seen in Table 2, where the novice proof writers' responses are shown by category, the most frequent response was *no* at 42.8%, with the majority of these students (67%) arguing that Statement A was the negation of Theorem 5. However, if the *yes* (28.6%) and *yes-no-yes* (23.8%) response categories are collapsed, then roughly half of the students were able to correctly respond to the validation statement. It is interesting to note that a secondary analysis of the sample's verbal responses, which coded students' responses for multiple instances of expressed hesitancy, equivocation, or doubt, found that 16 of the 21 students (76%) repeatedly articulated uncertainty. Lastly, interview participants overwhelming selected Theorem 5 for their "statement to prove."

Prompt 1: *You can prove Theorem 5 by proving Statement A.*		
Response	**N**	**%**
True (Yes)	6	28.6%
True (Yes-no-yes)	5	23.8%
False (No)	9	42.8%
Uncertain (Don't Know)	1	4.8%
Prompt 2: *If you were asked to prove Theorem 5, which would you pursue first?*		
Responses	**N**	**%**
Theorem 5	16	76.2
Statement A	2	9.5%
No Response	3	14.3%

Table 2. Novice Proof Writers' Theorem 5 Task Responses

Not surprisingly, the 6 mathematicians who were interviewed were, without exception, successful at the Theorem 5 validation task. Moreover, like the advanced students, their selection of a statement to prove was more balanced than that of the novices, with some expressing the selection "*either*." Data for the mathematician sample is provided in Table 3.

Furthermore, like the novices, 4 of the mathematician's (75%) expressed hesitancy or equivocations even though they did not shift their response to Prompt 1. However, unlike the novices, who doubted their responses, the mathematicians' expressions of uncertainty tended to be geared towards their own reasoning at the time of the interview; e.g., one remarked "I'm doubting myself right now for some reason" and another mentioned that it was "too early" in the morning for such questions.

Prompt 1: *You can prove Theorem 5 by proving Statement A.*		
Response	**N**	**%**
True	6	100
False	0	0
Prompt 2: *If you were asked to prove Theorem 5, which would you pursue first?*		
Responses	**N**	**%**
Theorem 5	2	33.3
Statement A	2	33.3
Either	2	33.3

Table 3. Mathematician's Theorem 5 Task Responses

Beyond the observed similarity of expressed hesitancy, the novices and mathematician responses were quite dissimilar. Indeed, while 5 of the 6 mathematicians approached the question of proving Theorem 5 by proving Statement A semantically, none of the novices were observed using a semantic approach. Instead, the novice proof writers tended to move away from the linguistic statements and work at a symbolic level, with many moving to truth tables to prove various equivalences. While it might seem that working at a symbolic level leads to a higher error rate, it appears that this is not the case. The same percentage of students who worked symbolically immediately responded *yes*, as did those who argued *no;* approximately 67% of each cohort. With this said, it is interesting to note that all (100%) of those who responded "*yes-no-yes*" worked symbolically. Since several students expressed a lack of comfort with the content as a rationale for moving to symbols, it may be that the mathematicians' greater content expertise played a role in their approaches to the Theorem 5 validation task. This finding raises questions about the skills novices need to evaluate statements with unfamiliar content.

A second important distinction between the novice proof writers' and mathematicians' approaches concerns how the two cohorts went about understanding Statement A. Specifically, the majority of novices (14 of 21; 67%) tended to engage in *chunking*; that is, they tended to break Statement A into two chunks, with the first containing the quantifying phrase, "there exists no positive integer," and the second chunk containing the open sentence, "$n \bmod(3) \equiv 2$ and n is a perfect square." Hence, when responding to Prompt 1, they sought relationships between the quantifying chunks of Statement A and Theorem 5 and then the open sentences, rather than holistically comparing the quantified statements, as illustrated in the transcript excerpt below.

Linda: This (Statement A) is the negation of this (Theorem 5) because this one says *for all* and this one says *there exists no* and this one is *if P then Q* and this one is *P and not Q*.

Figure 2. Chunking Transcript Excerpt

In contrast, the mathematicians and 4 of the successful students (3 who replied yes and 1 who responded *yes-no-yes*) engaged in *parsing*; that is, consideration of the various components of the quantified compound statements <u>and</u> their logical relations to each other. In other words, quantifiers were considered in relation to the open sentences they quantified rather than as separate sentence components. This approach is illustrated in the interview excerpt in Figure 3, where the student considers the negation of the quantifier in relation to the modified statement.

Nicolas: Because when you say "there exists no," that's a "for all" but then … but then you have to negate … (places fingers on P and ~ Q statements) … you, you have to negate Q.

Figure 3. Parsing Transcript Excerpt

Similarly, several mathematicians spoke aloud while responding to the Theorem 5 task, with many providing comments akin to the following mathematician's remark, "there exists no … so there is nothing that satisfies this (points to "$n \bmod(3) \equiv 2$ and n is a perfect square")."

Lastly, those who were observed *chunking* Statement A; that is, isolating the quantifying phrase "there exists no …" where also frequently observed becoming entangled in the natural language meanings and linguistic antonyms of "for all" and "there exists no." For instance, Patrick argued, as did others, that "for all" means "everything," "there exists no" means "nothing," and that "the opposite of nothing is everything." Thus, for some students, natural language functioned as an obstacle to validating the claim $S^* \rightarrow S$.

Discussion

The findings of the electronic survey of advanced mathematics students and the clinical interviews with novice proof writers and mathematicians indicate that while advanced students and mathematicians are quite successful validating the claim "You can prove Theorem 5 by proving Statement A," these determinations are quite difficult for novice proof writers who may engage in potentially unproductive chunking practices and employ inappropriate linkages between mathematical statements and natural language. While it is easy to argue that the majority of novices were simply weak in the content area of quantifiers and did not interpret them appropriately, for they neither engaged in valid parsing practices nor did they interpret the terms in a logically appropriate manner, there is reason to caution against this reaction. Many "introduction to proof" texts use natural language when introducing students to quantifiers and their negations. Moreover, in a review of these texts, it was found that none addressed the ambiguities that arise from natural language in relation to quantification (*cf.* Epp, 2003). Indeed, as was evident from a survey of dictionaries, and is illustrated in Figure 4, neither is there a transitive property for antonyms nor is there a lack of ambiguity to natural language when it comes to quantifying terms such as *for all* (everything), *exists no* (nothing) and *exists* (something; some), when expressed and negated using natural language antonyms.


eve·ry·thing
/ˈevrē ˌTHiNG/
pronoun
pronoun: everything
1. all things; all the things of a group or class.
"he taught me everything I know"
synonyms: each item, each thing, every single thing, the lot, the whole lot; More
antonyms: nothing

noth·ing
/ˈnəTHiNG/
pronoun
1. not anything; no single thing.
"I said nothing"
antonyms: something

2. existence (noun)
The state of being, existing, or occurring; beinghood.
Antonyms:
nothingness

2. some (pronoun)
A certain number, at least one.
Antonyms:
none, much, no, many
Synonyms:
a few

Figure 4. Natural Language Examples of Quantified Terms

Finally, the novices "statement to prove" selections, which indicated a strong preference for Theorem 5, stand in contrast to those of advanced students and mathematicians, with neither group demonstrating a preference. Given that many novices had difficulty parsing Statement A this result is not surprising. With this said it appears that, at least for novices, the metatheoretical issues described by Antonini and Mariotti (2008) may play a role in students' interpretations and sense of certainty in the context of the results of a proofs by contradiction. In particular, given that in response to Prompt 1, 42.8% of novices replied *false* and only 28.6% replied *true*, there is reason to believe that – as Antonini and Mariotti have argued – students experience difficulties at the metatheoretical level in relation to the theorem $S^* \rightarrow S$, with the current work indicating that a potential source of these difficulties may be validating the relationships between S* and S.

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