The complement of RUME: What's missing from our research?

Natasha Speer
University of Maine
speer@math.umaine.edu

Dave Kung
St. Mary's College of Maryland
dtkung@smcm.edu

The Research in Undergraduate Mathematics Education (RUME) community has generated a substantial literature base on student thinking about ideas in the undergraduate curriculum. However, not all topics in the curriculum have been the object of research. Reasons for this include the relatively young age of RUME work and the fact that research topics are not necessarily driven by the content of the undergraduate curriculum. What topics remain largely untouched? We give a preliminary analysis, with a particular focus on concepts in the standard calculus sequence. Uses for this kind of analysis of the literature base in the education of novice researchers and potential future directions for further analyses are discussed.

Key words: student thinking, calculus, literature review

Overview

The inherently applied nature of research in undergraduate mathematics education (RUME) means it is natural for non-researchers seeking information about teaching and learning to desire (or perhaps even expect) that members of the RUME community carry out research that spans the undergraduate curriculum. This desire and/or need for research-based information about how student learn particular content, vetted teaching practices, and validated assessment measures can prompt college mathematics instructors to ask of their mathematics education colleagues, "Is there any research on [topic X]" where "topic X" might be anything in the undergraduate curriculum from limits to infinite-dimensional vector spaces. Although our field is generating research related to many topics, we are far from having a complete catalog of findings on all topics taught in undergraduate mathematics courses.

In a field as young as RUME, it is to be expected that considerable effort is focused on the development and refinement of theory as researchers work to identify and characterize factors that shape learning. There is, however, value in periodically taking stock of where we are. As a contribution to this effort, we pursued the following questions: What topics from the undergraduate curriculum have been the objects of research with direct links to practice? In particular, if one looks through the table of contents for typical texts for the calculus sequence, which items are associated with research findings about student thinking and which have yet to be examined? In short, what's the complement of our existing literature base? In this Preliminary Report, we offer our answers to these questions based on literature reviews we conducted in the process of writing a book on student thinking for novice mathematics instructors. Our findings indicate that the contents of the complement include some topics known to be challenging for both students and instructors and some topics with strong connections to concepts from secondary mathematics. Although we provide a rationale for this work below and some findings in subsequent sections, we consider this work "preliminary" because we are seeking feedback on potential uses for and ways of representing/communicating the products of this kind of literature review.

Rationale and Relevant Literature

Others (e.g., Schoenfeld, 2007) have noted tensions that can exist between theory development and the desire to address practical or applied issues in fields such as education research. On the one hand, from economic and other societal perspectives there are significant and pressing needs to improve the teaching and learning of undergraduate mathematics (Holdren & Lander, 2012) and to do so we need research-based answers to questions derived from practice. On the other hand, the relative newness of the field of RUME means that we have significant needs for basic research and the associated development of theories. We see two primary ways in which an "inventory" of topics already addressed and missing can aid the field in balancing these needs and in making progress in ways that address both the applied and theory-focused needs in the field.

The first rationale is connected to the fact that we are not currently operating in a time of "normal science." Times of "normal science" (Kuhn, 1970) are characterized by having methods for conducting research in a field with substantial track records of accepted use as well as theory to guide investigations that has accumulated and withstood examination over time. Although there have been substantial advances in these areas over the past several decades, the field of RUME is still in a phase of significant theory-building and development of tools for research. Although this description characterizes the field overall, some areas have been the object of more research than others. Therefore, it is productive to identify areas in which the most development has occurred because those areas may be where methods and theory are most mature. This can help researchers identify theory and methods that might be good candidates for use when conducting studies in less well-developed areas in RUME.

The second reason an inventory could be useful has to do with the issue of problem identification. There are many reasons for selecting a focus for research and that decision need not be shaped only by where no research has yet been done. However, in seeking out areas for research, it can help to know where the gaps are. Informed decisions can then be made about whether questions of significance can be pursued by venturing into an area where literature is scarce or by focusing instead on issues in an already somewhat established area. This is of particular importance for those who mentor new researchers (e.g., graduate students) because "beginning researchers need to learn how to identify and frame workable problems– meaningful problems on which legitimate progress can be made in a reasonable amount of time" (Schoenfeld, 1999, p. 170). And unfortunately, "For most students, *problem identification* is not part of the research apprenticeship process" (Schoenfeld, 1999, p. 168).

Research Design

This is the story of what happened when we set out to write an instructors' guide to student thinking about topics spanning the undergraduate curriculum. Our goal was to provide novice instructors with insights into student thinking that were based on research findings and to do that for as many topics from the calculus sequence and core major courses as possible. To locate relevant literature, we did the following:

- Generated list of topics, basing it on a "standard" course sequence, with particular emphasis on the calculus sequence because of our primary target audience for the book (graduate students and novice faculty instructors). We consulted tables of contents from widely used textbooks (e.g., Stewart, 2015).
- Set out to find published research, utilizing Google Scholar and literature reviews or summaries (e.g., Carlson & Rasmussen, 2008). We created lists of relevant articles for each of the sections as part of the preparation for drafting chapters of the book.

- Sent draft chapters to expert scholars in the area of the chapter and asked if we were missing anything. In particular, we asked them: Have we faithfully reported the literature, remaining true to each researcher's intent while still making it accessible to those outside of the education world? Have we missed any significant findings or citations that should be included? We then revised chapters based on the feedback, incorporating additional references suggested by reviewers.
- We gave up on writing some chapters or sections because of lack of literature.
- After all chapters were drafted, reviewed, and revised, we returned to our original topic list. We compared that to the final table of contents and noted topics that were missing.

Our goal was to generate text that illustrated how students think (productively and unproductively) about key ideas in the undergraduate curriculum. Because of this particular focus, we were seeking research of a particular sort—research that either had an explicit focus on describing student thinking about the topic or where there were clear implications from the findings to students' ways of thinking. This focus means that certain kinds of work that are incredibly valuable to the RUME community (e.g., teaching studies, theory development and testing, assessment and evaluation studies) were outside the scope of our search.

Findings

For the purposes of this proposal, we present the findings based on our analysis of the literature review done for just a few chapters. In particular, we focus on topics in calculus: limit, derivative, application of derivatives, integral, and sequences and series. The references listed are representative of the literature we located for each topic – however, they are not comprehensive of all existing literature on the topic.

Limits

This is perhaps the most extensively covered topic in the calculus sequence. We found research findings related to both student thinking about core ideas of limit and computations done to determine limits. These findings include, for example, ways students think (incorrectly) about limits of functions by treating all functions as if they are continuous (e.g., Bezuidenhout, 2001; Tall & Vinner, 1981), thinking of a limit as an unreachable bound (e.g., Davis & Vinner, 1986; Grabiner, 1983), viewing all limits a monotonic (e.g., Davis & Vinner, 1986), and believing that testing a few values is sufficient to evaluate a limit (e.g., Williams, 1991).

Derivatives

We located research on student thinking about derivative as well as about computations to generate derivatives and uses of derivative on applied problems. This literature addressed various specific topics, including:

- student difficulties with the building blocks of derivative such as understanding the secant line representation for limits of difference quotients (e.g., Carlson, 1998; Habre & Abboud, 2006; Monk, 1994; Orton, 1983);
- difficulties with applying various procedures for calculating derivatives (e.g., Cipra, 2000; Horvath, 2008; Smith & Ferguson, 2004; Zandieh, 2000);
- trouble with graphical representations of derivatives (e.g., Ferrini-Mundy & Graham, 1994; Nemirovsky & Rubin, 1992; Orton, 1983);
- thinking associated with linking features of first and second derivatives (positive, negative, increasing, decreasing, etc.) with properties of the original function (e.g.,

increasing, decreasing) (Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, 2002; Zandieh, 2000).

What's missing from the literature on derivatives?

In comparing the sections of commonly-used texts to the list of topics for which we could locate research findings, several stood out as missing from the existing literature. The topics that are in the "complement" of the existing literature on derivatives include:

- implicit differentiation, in particular examinations of what sense students make in the transition from df/dx to d/dx and the idea of differentiation as an operator;
- student thinking and sense making about linear approximation and differentials;
- connections between trigonometric functions (as ratios of lengths of triangle sides) and the calculus of them;
- Newton's method, in particular what sense students make of the process.

Some topics that have been addressed in only a few studies could benefit from substantially more attention from researchers include: chain rule, related rates, and optimization.

Integrals

As with derivatives, the literature on student thinking about integrals provides insights into how students think about the concept of integration and how they think about computations used to determine the value of integrals. The research articles we located addressed various topics, including the following:

- definition and meaning of integrals (e.g., Abdul-Rahman, 2005; Fuster & Gómez, 1997; Gonzalez-Martin & Camacho, 2004);
- integral as a measure of accumulation (e.g., Anaya & Cavallaro, 2004; Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, 2002; Thompson & Silverman, 2008);
- Riemann Sums (e.g., Bezuidenhout & Olivier, 2000; Oehrtman, 2009; Thompson & Silverman, 2008);
- anti-derivative computations (e.g., Hirst, 2002; Orton, 1983b).

What's missing from the literature on integrals?

Substantial topics from a standard treatment of integration for which we were unable to locate research include:

- integration techniques involving the very commonly-used method of substitution;
- other integration techniques and what sense, if any, students make of this topic;
- volumes of revolution.

Sequences and Series

This very challenging topic has been the object of a fair amount of research on student thinking about the following:

- characteristics of the real numbers and how that intersects with understanding sequences and series (e.g., Anderson, Austin, Barnard, & Jagger, 1998);
- ideas related to limit and their impact on thinking about sequences and series (e.g., Li & Tall, 1993; Tall, 1977);
- issues of representation, in particular, coordinating algebraic and other views of sequences and series (Cornu, 1991; Tall & Vinner, 1981);
- definition of convergence and proofs of it (e.g., Harel & Sowder, 1998; Pinto & Tall,

1999).

What's missing from the literature on sequences and series?

Topics where research is very limited included:

- power series, especially the question of what sense students make of the overarching idea of approximating one function with other functions;
- Taylor and Maclaurin Series and what students think the core ideas are behind the computations we ask of them.

Conclusions, Implications and Ideas for Further Research

Efforts of the RUME community have resulted in research-based answers to the question of how students think productively and unproductively about a wide range of topics in the undergraduate curriculum. From our analysis of that literature and the topics typically addressed in that curriculum, there are ideas for which we still lack insights into student thinking. Looking just at the calculus sequence, limit appears to have been a rich and productive arena for research, resulting in findings about key ideas as well generating and refining theories about student thinking. For differentiation and integration, the literature does not seem to address the topics as uniformly. Although the existing research provides valuable insights into student thinking about some ideas, there are some noticeable gaps. These include the challenging ideas and techniques associated with implicit differentiation, linearization, various techniques of integration, and applications such as volumes of revolution. The community might also benefit from additional investigations into student thinking about power and other kinds of series. As with topics such as implicit differentiation and some techniques of integration, we suspect that there is much to be learned about what sense students do (and do not) make of the processes used in these sections of the course. The areas in which significant amounts of research have already occurred might be useful sources of theories that could be tested out in the less-researched areas.

Our search for literature was limited to research on student thinking and learning but further analyses could be conducted of research related to teaching, curriculum development or other genres of studies found in RUME. In addition, a separate kind of analysis could be done of just the works represented in RUME proceedings. This might provide insights into trends in the field or areas of investigation that have not yet found their way into journal publications.

Describing the complement of what exists in the literature may be useful in guiding graduate students or others new to the field. Knowing where theory development and findings are scarce or plentiful can help researchers (and those who advise them) to know whether their chosen topic is apt to take them into well-understood territory or whether they will encounter few studies and perhaps only limited theoretical frameworks to guide their efforts.

Questions to be posed to the audience:

- What are the reasons behind the dearth of research on various topics? Are these simply areas the last to be visited – or is there something about some topics that make them less interesting or productive for researchers?
- Might the RUME community benefit from this type of high-level view of its research – including who is studying what mathematical topics? What is the best way to assemble such information – and disseminate it?

References

Abdul-Rahman, S. (2005). Learning with examples and students' understanding of integration. In *Proceedings of the Eighth International Conference of Mathematics Education into the 21st Century Project on Reform, Revolution and Paradigm Shifts in Mathematics Education. Johor Bahru: UTM.*

Anaya, M., & Cavallaro, M. I. (2004). Visualization, Conceptual and Procedural Knowledge in Mathematics for Engineers. The Case of the Definite Integral. In *European Society for Engineering Education - Maths Working Group*.

Anderson, J., Austin, K., Barnard, T., & Jagger, J. (1998). Do third-year mathematics undergraduates know what they are supposed to know? *International Journal of Mathematical Education in Science and Technology*, *29*(3), 401–420.

Bezuidenhout, J. (2001). Limits and continuity: some conceptions of first-year students. *International Journal of Mathematics Education in Science and Technology*, *32*(4), 487–500.

Bezuidenhout, J., & Olivier, A. (2000). Students' conceptions of the integral. In *Proceedings of the Conference on the Psychology of Mathematics Education* (Vol. 2, pp. 2–73).

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *CBMS Issues in Mathematics Education: Research in Collegiate Mathematics Education III* (pp. 114–162).

Carlson, M., & Rasmussen, C. (2008). Making the connection: Research and teaching in undergraduate mathematics education. *MAA Notes*. Washington, DC: Mathematical Association of America.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378.

Cipra, B. A. (2000). *Misteaks... and How to Find Them Before the Teacher Does*. AK Peters, Ltd.

Cornu, B. (1991). Limits. *Advanced Mathematical Thinking*, 153–166.

Davis, R. B., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior*, *5*(3), 281–303.

Ferrini-Mundy, J., & Graham, K. (1994). Research in Calculus Learning: Understanding of Limits, Derivatives, and Integrals. In J. J. Kaput & E. Dubinsky (Eds.), *Research Issues in Undergraduate Mathematics Learning: Preliminary Analysis and Results* (Vol. 33, pp. 29–45). Washington, DC: The Mathematical Association of America.

Fuster, J. L. L., & Gómez, F. J. S. (1997). Una Interpretaci{ó}n de las Dificultades en el Aprendizaje del Concepto de Integral. *Divulgaciones Matem{á}ticas*, *5*(1/2), 61–76.

Gonzalez-Martin, A. S., & Camacho, M. (2004). What is first-year Mathematics students actual knowledge about improper integrals? *International Journal of Mathematical Education in Science and Technology*, *35*(1), 73–89.

Grabiner, J. V. (1983). Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus. *The American Mathematical Monthly*, *90*(3), 185–194.

Habre, S., & Abboud, M. (2006). Students' conceptual understanding of a function and its derivative in an experimental calculus course. *The Journal of Mathematical Behavior*, *25*(1), 57–72.

Harel, G., & Sowder, L. (1998). Students' Proof Schemes: Results from exploratory studies. *Research in Collegiate Mathematics Education*, *3*, 234–283.

Hirst, K. (2002). Classifying Students' Mistakes in Calculus. In *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level)*.

Holdren, J. P., & Lander, E. S. (2012). *Engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics*. Washington, D.C.

Horvath, A. (2008). Looking at calculus students' understanding from the inside-out: The relationship between the chain rule and function composition. In *Eleventh Conference on Research in Undergraduate Mathematics Education (RUME)*.

Kuhn, T. (1970). *The Structure of Scientific Revolutions* (2nd ed.). Chicago: University of Chicago Press.

Li, L., & Tall, D. (1993). Constructing different concept images of sequences and limits by programming. *Proceedings of PME*, *17*, 41–48.

Monk, S. (1994). Students' understanding of functions in calculus courses. *Humanistic Mathematics Network Journal*, *27*, 21–27.

Nemirovsky, R., & Rubin, A. (1992). Students' Tendency to Assume Resemblances between a Function and Its Derivative. *The Working Paper Series*. Cambridge, MA: TERC Communications.

Oehrtman, M. (2009). Collapsing Dimensions, Physical Limitation, and Other Student Metaphors for Limit Concepts. *Journal for Research in Mathematics Education*, *40*(4), 396–426.

Orton, A. (1983a). Students' understanding of differentiation. *Educational Studies in Mathematics*, *14*(3), 235–250.

Orton, A. (1983b). Students' understanding of integration. *Educational Studies in Mathematics*, *14*(1), 1–18.

Pinto, M., & Tall, D. (1999). Student Constructions of Formal Theory: Giving and extracting meaning. *Proceedings of the 23rd Conference for the International Group for the Psychology of Mathematics Education*, *3*, 281–288.

Schoenfeld, A. H. (1999). The core, the canon, and the development of research skills: Issues in the preparation of education researchers. In E. C. Lagemann & L. S. Shulman (Eds.), *Issues in education research: Problems and possibilities* (pp. 166–202). San Francisco: Jossey-Bass.

Schoenfeld, A. H. (2007). Method. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 69–110). New York: MacMillan.

Smith, G. G., & Ferguson, D. (2004). Diagrams and math notation in e-learning: Growing pains of a new generation. *International Journal of Mathematical Education in Science and Technology*, *35*(5), 681–695.

Stewart, J. (2015). *Calculus: early transcendentals*. San Francisco, CA: Cengage Learning.

Tall, D. (1977). Cognitive conflict and the learning of mathematics. *International Group for the Psychology of Mathematics Education, Utrecht, Holland*.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151–169.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 43–52). Washington D.C.: Mathematical Association of America.

Williams, S. R. (1991). Models of Limit Held by College Calculus Students. *Journal for Research in Mathematics Education*, *22*(3), 219–236.

Zandieh, M. J. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *CBMS Issues in Mathematics: Research in Collegiate Mathematics Education* (Vol. IV(8), pp. 103–127).