Interpreting proof feedback: Do our students know what we're saying?

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Instructors often write feedback on students' proofs even if there is no expectation for the students to revise and resubmit the work. However, it is not known what students do with that feedback or if they understand the professor's intentions. To this end, we asked eight advanced mathematics undergraduates to respond to professor comments on four written proofs by interpreting and implementing the comments. We analyzed the student's responses through the lenses of communities of practice and legitimate peripheral participation. This paper presents the analysis of the responses from one proof.

Keywords: Proof Writing, Proof Grading, Proof Instruction, Proof Revision, Student Thinking

Introduction and Research Questions

Rav (1999) claimed proofs "are the heart of mathematics" and play an "intricate role … in generating mathematical knowledge and understanding," (p. 6). Proof is perhaps the dominant feature of the advanced undergraduate mathematics curriculum. While practice writing proofs is certainly important in developing students' proficiency with proof-writing, without feedback students are unlikely to improve their proof-writing. Moreover, mathematicians act as if they believe that giving students feedback is critical to their learning, writing marks and notes on student proof-productions (Brown & Michel, 2010; Moore, 2014; Strickland & Rand, in press). Yet this feedback improves student learning only if students read, make sense of, and incorporate it into their future work. But few studies have examined the effectiveness of this process of giving feedback and asking students to revise their proofs. Thus, in this study we investigate the following questions:

1. How do students interpret professors' marks and comments on student-written proofs?
    a. How do students interpret and describe each mark or comment?
    b. How do students explain the rationale for making the proposed changes?
2. What changes do students make to the proofs in response to their interpretations of the comments?
3. How do students' responses to each of questions 1a and 1b above align with the way that is normative in the discipline (as described by mathematically enculturated individuals)?

Literature and Theory

Theoretical orientation

The theoretical orientation for this study is a version of social constructivism referred to as the emergent perspective (Cobb & Bauersfeld, 1995; Cobb & Yackel, 1996), drawing on ideas primarily from the social perspective. In particular, "The social perspective indicated is concerned with ways of acting, reasoning, and arguing that are normative in a classroom community" (Cobb, et al, 2001, p. 118). Students produce proofs as part of class, and the professor holds them to some standard and, we argue, communicates the normative ways of reasoning and arguing for the classroom via written comments on their proofs. That is, within the class she acts as the representative of the mathematical community with a goal of helping students to develop discipline-specific ways of writing. As a result, we draw on the notions of communities of practice and legitimate peripheral participation (Lave & Wenger, 1991).

We take a community of practice to be a group of people coming together in a process of collective learning in a shared domain, in this case learning about advanced undergraduate mathematics. There are two overlapping communities of practice of import to this study: the community of professional mathematicians and the community of advanced undergraduate mathematics students where the professor acts as a representative of the community of mathematicians. In such situations, we take learning to be a move towards fuller participation in the practices of the target community. Legitimate peripheral participation is a way to understand novices' attempts to participate as well as the means by which they learn. While to a mathematical expert, novice proof-productions are filled with errors of logic, grammar, syntax and more, under the lens of legitimate peripheral participation we instead view them as attempts to communicate in the style of the community where the rules are, at best, partially mastered and often tacit.

Student and mathematician misunderstandings

While we argue that professor comments on student proof-productions have a significant role in student learning to produce proofs, we also have reason to believe that students are likely to misinterpret them. In particular, regarding conceptions of proof, previous research, including that of Ko and Knuth (2013) and Selden and Selden (2003) have shown that students often fixate on the form, such as the ritualistic inclusion of particular features or the presence of mathematical symbols, rather than the content of the proof.

Research on student understanding of lectures suggests that students develop significantly different understandings of the presented material and meaning for professor actions than the professor intends (cf. Lew, Fukawa-Connelly, Mejia-Ramos, & Weber, in press; Weinberg, Weisner, & Fukawa-Connelly, 2014). We use this prior research on misunderstandings and "misses of understanding" to form hypotheses about how students are likely to interpret a professor's comments on proof-productions. In particular, we hypothesize that they are likely to:

- not apprehend some comments,
- develop only a surface-level understandings of comments, and
- interpret comments in ways that differ from what mathematical experts would do.

Moreover, we argue that the latter two of these hypotheses are supported by the construct of legitimate peripheral participation as described above because learners are likely to make exactly these kinds of production and interpretation mistakes.

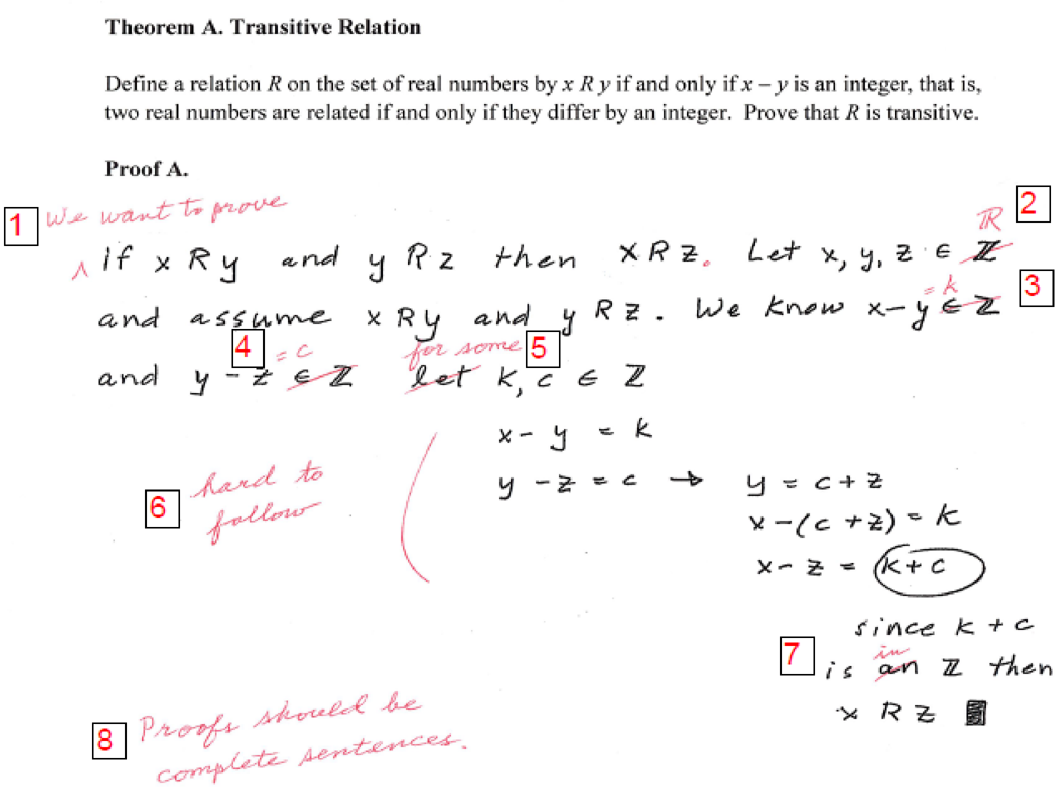
Methods

Participant selection

The participants were 8 students, 4 men and 4 women, with advanced undergraduate standing from two teaching-focused institutions, four from each institution. They had each taken at least 2 proof-based undergraduate mathematics classes, including a transition-to-proof course. We purposefully selected participants who had experience with writing proofs and receiving feedback from their professors so as to give the best possible chances for their success in understanding the proofs and interpreting the professor's comments in this study.

Data collection

We engaged each participant in a 90-minute task-based interview where the primary task was to describe and interpret a professor's comments about proofs. The interviews were audio-recorded and pencast with Livescribe pens. In each interview, we asked basic demographic questions and reflective questions about the participant's typical use of professor comments. Subsequently, we asked the participant to engage in the proof-comments task on a maximum of 4 proofs. We ensured that each proof had a mix of comments related to notation and presentation, and that some of the proofs also included comments that addressed logical issues. The proofs and professor comments were taken from a previous research project exploring professors' proof grading (Moore, 2014). An example proof, with comments, is shown in Figure 1.



Theorem A. Transitive Relation

Define a relation R on the set of real numbers by $x\,R\,y$ if and only if $x - y$ is an integer, that is, two real numbers are related if and only if they differ by an integer. Prove that R is transitive.

Proof A.

1 We want to prove

If $x R y$ and $y R z$ then $x R z$. Let $x, y, z \in$ ~~$\mathbb{Z}$~~ $\mathbb{R}$ 2

and assume $x R y$ and $y R z$. We know $x - y$ ~~$\in \mathbb{Z}$~~ $= k$ 3

and $y - z$ 4 $= c$ ~~$\in \mathbb{Z}$~~ for some 5 ~~let~~ $k, c \in \mathbb{Z}$

$x - y = k$

$y - z = c \rightarrow y = c + z$

$x - (c + z) = k$

$x - z = k + c$

6 hard to follow

since $k + c$

7 is ~~an~~ in $\mathbb{Z}$ then

$x R z$ ∎

8 Proofs should be complete sentences.

Figure 1. An example of the proofs and professor comments used in the interviews

We presented the written proofs to the participant one at a time, told her it had been written by a student, and asked her to read and understand it as best she could. Then we presented a marked proof to the participant with a professor's feedback written in red ink. To determine whether the participant's interpretation of the mark or comment matched our own, we asked the participant to explain why the professor had made each individual mark or comment and what changes she thought the professor wanted. Finally, we asked the participant to rewrite the proof in order to allow us to further explore her interpretation of the comments and see how she implemented the professor's recommendations.

Data analysis

For each comment in each proof each of the researchers wrote a description of what change we believed the professor wanted in the proof and an explanation for the change. Based on these individual notes, we created a consensus description of what each proof-comment was asking the participant to change and the reason for the change.

For each interview we first transcribed the interview and then chunked it at a number of levels. We parsed the demographic and reflective questions in one piece and the participant's discussion of each of proof in additional pieces. We partitioned the discussion of each proof by identifying the participant's initial reading and the beginning- and endpoints of their conversation about individual comments. In cases where participants discussed multiple comments in the same utterance, we looked across interviews, and when it was common, we treated the comments as a single unit to parse all interviews similarly. We made a final block of the talk-aloud proof-writing process, for which we chunked their utterances around the comments and linked those to what they were writing in the proof.

To code the participant's utterances about each comment we first wrote a brief holistic summary. Then we developed a coding sheet identifying:

- What the participant identified as the part of the proof the comment was addressing,
- What the participant's response suggests should change in the proof,
- Any reasoning that the participant gave to explain the intended change,
- An inference about the participant's thinking about proofs,
- A summary of what the participant changed in his proof revision,
- A comparison of each of the above to our consensus expert-interpretation, and
- An explanation of how an unanticipated change exhibited during the proof writing could be understood as a logical interpretation of the professor's comments.

We then created summaries first by summarizing across participants within individual proof-comments and then by further aggregating within types of comments (e.g., logical issues) and describing the understanding of proof that was demonstrated by the related responses.

Results

We report three principal findings in response to the research questions. In the interest of space, we will limit our presentation of evidence to the proof displayed above. Overall, the participants were very successful at interpreting what a professor wanted them to do in response to any comment. We saw the following success rates: For each of the eight comments 100% of participants correctly identified an acceptable part of the proof to be changed, and they all executed a change in a manner logically consistent with our understanding of the comment. However, how they interpreted the comments and executed the requested changes was not always consistent with our understanding. In the sections that follow, we explore the participants' work and thinking about the professor's comments.

Students' revisions of specified changes

Six of the professor's eight comments on Proof A were very specific. Five of them indicated to the student that something in the proof should be crossed out and replaced. For example, comment 2 in the first line of the proof in Figure 1 specifies replacing $\boldsymbol{Z}$ with $\boldsymbol{R}$. In these instances, the participants unsurprisingly always identified what they believed the professor wanted them to revise and implemented the revisions in a way that conformed with expert understanding. Comment 1 was also specific but suggested the addition of new text, namely the phrase "We want to prove," rather than the replacement of existing text. In this case, seven of the participants added the recommended phrase as indicated, whereas one participant, Adam, showed some individuality by assuming that xRy and yRz and then writing, "We will prove that xRz." Thus, when the professor's comment was either a change of proof-text or an addition to the proof-text and the change was explicitly written out, the participants' identification and implementations of the recommended changes were largely consistent with the expert consensus, but according to the expert consensus, the participants did not always understand why the professor recommended a change.

Students' interpretations of the logic of specified changes

The participants were also asked to describe why they thought the professor had specified these changes to the proof-text, and their answers revealed a variety of ways of thinking about proof. With 8 participants and 8 professor comments on Proof A, there were 64 opportunities for students to explain their reasoning about the comments. Of these 64 potential explanations, 41 agreed with the experts' consensus interpretation, 18 did not agree (partial agreement was counted as half agreement, half disagreement), one student simply did not give explanations for 2 changes, and 3 explanations were unclear. Three students gave 7 or 8 explanations that aligned with the expert interpretation, suggesting they were very strong students in terms of proof comprehension. The difference between "let" and "for some" was where students most commonly gave explanations that did not align with the expert consensus.

For example, consider the direction to add, "We want to prove" to the beginning of the proof. The expert consensus was that the statement improved the readability of the proof by making clear that the first sentence expressed the goal of the proof. The experts all agreed that the clause the student wrote could have been deleted, but as written, the statement assumes the conclusion of the theorem. Bella explained that she understood the reason for the comment as, "that's just one of the proper ways to start a proof, that from what I've learned, yeah, it's just the way to start a proof." That is, her thinking appears to focus on the form of proofs, that they may begin with "we want to prove" rather than focusing on the function of the statement. This type of form thinking was relatively rare among the participants.

Five participants described the added phrase as entirely focused on clarifying the presentation. For example, Don said, "This is just to me good syntax. It's a way of setting it up to be understand better and to be read more easily." Yet our expert consensus was that the professor intended to point out a lapse in logic, as noted by two participants, including Genevieve, who explained: "I think the professor meant that at the beginning of this [proof] you have this statement which is, uh, it's the claim that you are seeking to prove at the end, and so if you are assuming that every statement in the proof is true, having said it, you ought to have something at the beginning that says we want to prove it." Thus, we suggest that the logic the professor intended to motivate by this comment was not successfully communicated to six of the participants. But at the same time, we agree that the first sentence is unnecessary for a successful proof, i.e., it could be omitted. Note that in terms of the revised proof-productions, it would be impossible to distinguish between a participant who included the phrase "we want to show" because "that's how proofs should start" from one who did so for reader clarity, or from one who understood the logical underpinnings.

The participants also initially showed mixed understanding of the reasons for the "cross out and replace" comments that the professor wrote, and there were differences in the participants' interpretations when the comments were logically necessary as opposed to more stylistic. For example, for comment #2, which specified changing $\boldsymbol{Z}$ to $\boldsymbol{R}$, the experts argued that the student's proof-text did not actually prove the statement for all real numbers *x*, *y*, and *z*, and thus the change was logically necessary. Six of the interviewed participants gave an explanation that approached that of the experts, including Genevieve who said, "there is no reason to believe that *x*, *y*, *z* are in the integers. The theorem never states that they are in the integers. [The theorem states] on the set of real numbers." Don gave a somewhat mixed explanation, initially saying "they [$\boldsymbol{Z}$ and $\boldsymbol{R}$] are both correct but the real numbers are more applicable, in most cases," but later in the interview noting that the theorem specifies that *x*, *y*, *z* are real numbers. In this case, we suggest that the fact that the statement of the theorem invoked both real numbers and integers increased the difficulty of parsing the domains and relating them to the appropriate piece of the definition of the relation, and as a result, Don's explanation for the requested change did not reject the original statement as inappropriate for the proof. Finally, we note that most participants gave a response that described normatively correct logic, yet none of them noted that the original proof attempt did not prove the theorem.

The changes requested in comments 3 and 4 were to change $x - y \in \boldsymbol{Z}$ and $y - z \in \boldsymbol{Z}$ to $x - y = k$ and $y - z = c$. The expert consensus interpretation describing the rationale for this comment was that the professor was attempting to give advice that would help the student revise the proof as written, rather than changing the structure of the subsequent argument. Thus, assuming that the remainder of the argument was to be preserved as much as possible, and because the student was asked to let $x - y = k$ and $y - z = c$ in that argument, the expert consensus was that this change is stylistic. As the statements are written originally, while possibly confusing due to lack of parentheses, they are correct and contribute to the argument, but they induce redundancies in the subsequent algebraic part of the argument.

Charles noted, as did five other participants, that "the student failed to define *k* and *c*, in my opinion, as necessary constants. It was a bit ambiguous…" Thus, these participants seemed to recognize that the professor's comment is directed at a logical issue, how constants should be introduced and defined. These participants wanted the constants to be defined earlier and specifically identified as being members of a particular domain.

In reference to comment 5, only two students, Don and Nancy, articulated the distinction between "let" and "for some," and only Genevieve gave a reason in her revised proof for the changes corresponding to comments 3, 4, and 5 by referring to the definition of the relation *R*. Thus, again we argue that the participants did not fully understand the motivation for the professor's comment.

Comments that did not specify the change

Two comments on Proof A were more general in that they did not ask for specific changes. Comment 6 was "hard to follow" and comment 8 was "proofs should be complete sentences." As for the first of these, the participants agreed with us that the main issue was that the algebraic steps lacked readability, and they succeeded in writing revised proofs that were more readable. In response to comment 8, four participants rewrote the entire proof in complete sentences, including the sequence of algebraic equations, whereas the other four displayed a sequence of algebraic equations. Both are reasonable interpretations of the professor's note and are stylistically acceptable.

Conclusion

The first significant aspect of this study is that it is the first study that describes and analyzes how students interpret and respond to the comments that professors write on proofs. We recognize that this is a single, exploratory case study with only analytical generalizations. Moreover, we note two significant limitations to this study that suggest the need for further work in this area. First, the participants were reading and writing proofs on mathematical topics that most of them had not worked with in some time, possibly since their introduction to proof class.

The second significant limitation is that we asked the participants to interpret comments on proofs they had not written, thus imposing a need to make sense of another student's proof attempt prior to interpreting the comments. These limitations raise questions of how students' ability to interpret comments relates to their proof-writing and proof-comprehension abilities generally. More research is clearly needed to explore these questions, yet without a body of empirical evidence, there is no basis for more theoretical work. Thus this first exploratory study provides direction for those further studies. We note one final limitation: initially the four experts did not always agree on the reasons for the changes. While we could come to a consensus interpretation, there were significant differences in our initial interpretations, which means that different researchers, or a different mix of researchers, might have arrived at a different consensus interpretation of the professor's comments. This is a limitation of the study and suggest an avenue for future research, motivated by Weber's (2014) argument that proof is a clustered concept. We hypothesize that while professors might share instructional goals about proof and use similar notes and language to communicate with students, in reality they may be attempting to convey very different content via the same notes, which has significant implications for students.

The first finding is that when participants revised the written proofs, they made all of the changes requested by the professor and very few changed anything that was not requested. In particular, when the professor specifically indicated a change in the way to write the proof, such as replacing a symbol or adding a phrase, most participants made the exact change that the professor suggested. When the professor's comment was more ambiguous, such as "write in complete sentences," the participants all complied with the request, but interpreted it in different ways, some writing a paragraph proof with equations in sentence form, and some writing a stack of displayed equations but adding text at more strategic points. Similarly, the phrase "hard to follow" prompted some participants to include reasons for each step in the revision while others made more minimal revisions. Only two participants revised a portion of the proof that the professor did not specifically indicate: two rewrote the beginning of the proof and one added a reason that the integers k and c must exist by the definition of the relation R.

The second finding is that the participants often gave explanations for the requested changes that did not align with how experts understood the reason for the changes. The participants were more likely to over-attribute the notion of "sounds better" or "that's what you do," which we interpreted as describing the cultural conventions of proof.

The third finding, which is closely tied to the second, reinforces the claim that participants often fail to understand professors' lectures in the intended way, or even in the way that mathematical experts do. Although the participants made the requested changes, they missed the professor's reasoning because the professor's comments did not convey the difference between logically necessary and stylistic changes, nor did the comments help the participants understand the logically problematic aspects of the original proof.

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