Students' concept image of tangent lines compared to their understanding of the definition of the derivative

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Our research explores first-semester calculus students' understanding of tangent lines and the derivative concept through a series of three interviews conducted over the course of one semester. Using a combination of Zandieh's (2000) derivative framework and Tall and Vinner's (1981) notions of concept image and concept definition, our analysis examines the role that students' concept image of tangent lines plays in their conceptual understanding of the derivative concept. Preliminary results seem to indicate that students are more successful when their concept image of tangent includes the limiting position of secant lines, as opposed to a tangent line as the line that touches the curve at one point.

Key Words: Tangent Line, Derivative, Conceptual Understanding

"Conceptually, the role of visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject" (Zimmerman, 1991, p. 136). This quote encapsulates the relevance and motivation of our research efforts in studying students' understanding of tangent lines in first-semester calculus. Given the crucial role that tangent lines play in the visual aspects of the derivative concept, it is pertinent to consider how misconceptions about tangent lines may contribute to a lack of conceptual understanding of the derivative concept. Our preliminary analysis seems to reveal that students who consistently defined a tangent line as the limiting position of secant lines were also able to graphically explain the definition of the derivative, while those who used other definitions for a tangent line, such as a line intersecting the graph at only one point, were not able to do so.

Literature Review

It has been well documented that students' early experiences with tangency in geometry have the potential to negatively affect their understanding of tangency in subsequent settings (Fischbein, 1987; Tall, 1987; Winicki & Leikin, 2000; Biza, Christou, & Zachariades, 2008). In addition, Biza's (2011) study demonstrated that developing a concept definition of tangency characterized by its use in analysis is both difficult and non-intuitive for students. Vincent, LaRue, Sealey, and Engelke (2014) identified misconceptions first-semester calculus students may have concerning tangent lines, such as believing that several tangent lines may exist at a single point or confusing the notion of a tangent line with the tangent function ($y = \tan x$).

We know that students typically prefer working with the derivative concept algebraically and often struggle with the visual aspects (Asiala, Cottrill, Dubinksky, & Schwingendor, 1997; Habre & Abboud, 2006). Similarly, Hahkionieme's (2006) research project accentuated the difficulties students have interpreting what the formal definition of the derivative means graphically. Other studies have highlighted students' confusions concerning the relationship between tangent lines and the derivative, such as confusing the y-coordinate of the point of tangency with the derivative or equating the tangent line to the derivative (Orton, 1983; Amit & Vinner, 1990).

Theoretical Perspective

Our research incorporates a blend of two theoretical perspectives: Tall and Vinner's (1981) *concept image* and *concept definition* and Zandieh's (2000) derivative framework. The *concept image* represents the overall cognitive structure constructed by the learner and includes all the mental pictures and associated properties and processes that an individual has built up over time. Any *concept image* has a related *concept definition,* which is a learner's description of his or her understanding. Based on a student's language and written work, interpretations can be made about pieces of knowledge that do or do not belong to the concept image.

According to Sfard (1991), *processes* are operations on previously established objects that can be *reified* into *objects* and acted on by other processes. This forms what Zandieh (2000) calls *process-object* pairs. Zandieh's derivative framework (Table 1) is made up of three layers (Ratio, Limit, and Function) of process-object pairs and is meant to describe what the mathematical community means by the concept of the derivative (within four key contexts) at the first-year calculus level. Part of an individual's understanding may be noted within the grid when he or she mentions a context and any component of the layers of the derivative concept in response to interview questions. A circle in the grid indicates that the student has at least demonstrated a pseudo-object (an object with no internal structure) understanding of the row and column that intersect at that box. A shaded circle represents that the student has also demonstrated an understanding of the underlying process of the layer.

Process-object layer	Graphical (Slope)	Verbal (Rate)	Physical (Velocity)	Symbolic (Diff. Quotient)	Other
Ratio					
Limit					
Function					

Table 1. Zandieh's derivative framework.

Methodology

Our study took place during the spring 2015 semester with twelve first-semester calculus students enrolled in a large public university. Each participant completed a series of three interviews over the course of one semester: beginning half of the semester (immediately following instruction on tangent lines and the definition of the derivative), midterm, and end of term. Each interview focused on the concept of tangent lines- students' personal concept definition as well as tasks involving construction of tangent lines. Interviews 2 and 3 additionally consisted of tasks involving the derivative- sketching the graph of $f'(x)$ given the graph of $f(x)$ and interpreting the formal definition of the derivative graphically.

We are currently in the process of analyzing the data. At this stage, we have completed a detailed analysis for two of the twelve participants (Jamie and Andy) and a surface level analysis of the remaining ten. Jamie and Andy's series of interviews have undergone multiple viewings along with detailed notes and have been transcribed and coded. To code the data related to the derivative concept, we used Zandieh's derivative framework (Table 1, above). The concept of tangent lines is situated within the first two layers of this framework, and so, a modified version of the framework (Table 2) was used to code interview data related to tangent lines. Comparing these coded sections of the data, we are interested in examining relationships between students' concept images of tangent lines and the derivative.

Process-object layer	Graphical (Slope)	Verbal (Rate)	Physical (Velocity)	Symbolic (Diff. Quotient)	Other
Ratio					
Limit					

Table 2. Modified version of derivative framework used to code data on tangent lines.

Results

The results presented in this section focus on the preliminary analysis of one of twelve of the participants (Jamie), but we will also make reference to some of the other participants as well as discuss general themes found in the data, thus far. We will specifically focus on analysis of Jamie's concept image of tangent line (Interviews 1, 2, and 3) and her graphical understanding of the definition of the derivative (Interview 3), identifying relationships between the two.

Throughout all three interviews Jamie consistently defined a tangent line in terms of its “one point” relationship with the graph. When constructing tangent lines and justifying her work, she mainly reasoned about the location of the point of tangency and whether or not the tangent line should be “above” or “below” the curve. She never defined a tangent line in terms of the limiting position of secant lines. Due to her unstable concept definition, she almost always constructed a tangent line at places where one should not have existed. Table 3 shows Jamie's coded chart for her responses to the question “What is a tangent line?” from Interviews 1, 2, and 3.

Process-object layer	Graphical (Slope)	Verbal (Rate)	Physical (Velocity)	Symbolic (Diff. Quotient)	Other
Ratio		○			
Limit	○				

Table3. Jamie’s summary chart for definition of tangent line.

The open circle in the limit row represents Jamie's definition of a tangent line as “where it hits one point on a graph.” Since she did not discuss the limiting process, this circle is not shaded in, and since she did not mention the notion of slope in any of her explanations, there is not a code in the slope row.

During Interview 3 Jamie was asked to graphically interpret the formal definition of the derivative concept. Table 4a below shows an excerpt from her transcript.

199	Jamie	Maybe like when you graph it you determine the tangent line. Maybe if you want to find like one exact point I guess on it. I don't know.
200	Int.	Ok. So if you come back to this for a second. So, you're not sure about, like if you take the limit part away do you know what this portion of the definition of the derivative represents?
201	Jamie	No. No.
205	Jamie	Oh! The *h* might be a slope of zero.
206	Int.	So, what do you mean by that?

207	Jamie	Like, here it'd be zero and here would be zeroes [constructs horizontal tangents].

Table 4a. Jamie. Example response. Definition of derivative

Jamie did not discuss the role of tangent lines on her own initiative. Her ideas in line 199 were a response to the interviewer's question about the role, if any, tangent lines may play in the graphical interpretation of the derivative. She mentioned the idea of finding "one exact point," and this response was coded with an open circle in the limit row (Table 4b). Jamie was also uncertain about the meaning of h in the definition of the derivative, and because h is going to zero, the idea of zero slope (horizontal tangents) was triggered in her concept image. The open circle in the ratio row demonstrates her understanding that slope is somehow involved in the derivative concept, but this circle is not filled in because again, she does not discuss the process of determining slope, such as rise over run or change in y over change in x. Table 4b shows Jamie's coded chart for her entire response to the graphical derivative question posed during interview 3.

Process-object layer	Graphical (Slope)	Verbal (Rate)	Physical (Velocity)	Symbolic (Diff. Quotient)	Other
Ratio	○			x	
Limit	○			x	
Function				x	

Table 4b. Jamie's summary chart for definition of the derivative.

It is important to note that although Jamie did not mention the notions of rate of change or velocity during her explanations, other students did. Since participants were given the symbolic definition of the derivative and were not responsible for generating it on their own, we were unable to code the symbolic column of the chart and have labeled it with x's.

Although Jamie's concept image for tangent line and her concept image for derivative have similar structures and seem to influence one another, throughout the interviews she demonstrated that she was unaware of such connections (Table 5). For example, she often could not mathematically justify her work, stating: "it's not coming to me yet", or "cause this is how we learned it in class", or simply "I don't know". She also reasoned that it was possible to contruct tangent lines at places where the derivative didn't exist. So, even though we see similar structures in Jamie's concept images according to Zandieh's framework, she is not aware of the connection between these to concepts.

216	Int.	Ok. Um, so you said that this is the definition of the derivative and then I asked you about tangent lines and their relationship and what was your... what do you think about that?
217	Jamie	Maybe there is. Maybe there isn't.
196	Int.	Ok. And are tangent lines...how do tangent lines...what role do they play in this whole thing?
197	Jamie	I don't know.

Table 5. Example of disconnect between concept images.

Considering Tables 3 and 4b, we see that pseudo-objects in Jamie's concept image of tangent line transferred to pseudo-objects in Jamie's concept image of the graphical derivative (or vice versa). We see similar results with Andy (Table 6)- structural understandings in the layers of tangent line transferred to structural understanding in the layers of the graphical derivative.

	Definition of Derivative		Tangent Line	
Process-object layer	Graphical (Slope)	Verbal (Rate)	Graphical (Slope)	Verbal (Rate)
Ratio	●		●	○
Limit	●		●	
Function	○		N/A	N/A

Table 6. Andy*'s s*ummary *c*hart *of the d*efinition *of the d*erivative *and tangent lines*.

These preliminary results reveal a connection between students' personal concept definition of tangent line and their graphical understanding of the derivative concept. In reviewing the data of all twelve participants, only two participants were able to graphically explain the definition of the derivative. The remaining ten were unsuccessful in their attempts and also consistently (in their definitions and justifications) used other definitions for tangent such as a line intersecting the graph at only point and did not use the limiting position of secant lines definition**.**

In comparing Jamie to Andy, we see Andy exhibited an understanding of the processes involved in the layers of both concepts. Andy often indicated he was "searching" his concept definition, using phrases such as, "hold on let me think about this." Jamie never demonstrated such activity. In contrast, she exhibited that her reasoning mainly flowed from her concept image, using phrases such as "this is how we did it in class" or referencing "similar homework problems". Vinner (1991) discussed that students most often reason from their concept image rather than their concept definition. We are interested in further exploring these ideas and their implications as we progress in our analysis.

Our preliminary results indicate that students' understanding of the graphical derivative may be strongly influenced by their concept definition of tangent lines. While this connection may not be surprising, it *is* surprising how many tasks Jamie was able to complete with her "one point" concept definition of tangent line. Even though she was able to sketch most tangent lines accurately, she was never able to connect the definition of the derivative with the tangent line. Andy, however, was able to make these connections, and our preliminary analysis suggests that his concept image of tangent line as the limiting position of secant lines played a big part in his success. Additional data analysis is necessary to further explore this conjecture.

Implications for teaching

Shortcut definitions for tangency, such as "the line touching the graph at one point" or even "the line whose slope is equal to the derivative" are helpful but should not replace the definition of tangency as the limiting position of secant lines. Consistent classroom use of shortcut definitions may result in the creation of pseudo objects within students' concept images of tangent line and the derivative. These definitions do not accentuate the underlying processes involved in the layers of Zandieh's framework. We do not imply that consistent use of the secant line definition of tangency will magically result in structural understandings, but our results do seem to imply that pseudo objects within the tangent line concept image transfer over to the derivative concept image, and likewise for structural understandings.

Questions for the audience

1. What are your thoughts on the modified version of the derivative framework for tangent lines?
2. What role should the *series* of interviews play in data analysis?

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