ANALYZING STUDENTS' INTERPRETATIONS OF THE DEFINITE INTEGRAL AS CONCEPT PROJECTIONS

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This study of beginning and upper-level undergraduate physics students extends earlier research on students' interpretations of the definite integral. Using Wagner's (2006) transfer-in-pieces framework and the notion of a concept projection, fine-grained analyses of students' understandings of the definite integral reveal a greater variety and sophistication in some students' use of integration than previous researchers have reported. The dual purpose of this work is to demonstrate and develop the utility of concept projections as a means of investigating knowledge transfer, and to critique and build on the existing literature on students' conceptions of integration.

Key words: Definite integral, Knowledge transfer, Physics, Knowledge in pieces

This article, rooted in Wagner's (2006) transfer-in-pieces framework, considers the problem of knowledge transfer from mathematics into physics, although the implications extend to other disciplines as well. A distinguishing characteristic of this perspective is that knowledge flexibility and transfer at all levels of expertise are supported not by a purely abstract quality of the knowledge in question, but by its ability to adapt to and accommodate contextual differences. In this sense, knowledge is said to be context sensitive. Wagner (2006, 2010) argued that applying a single mathematical principle or concept across a variety of contexts, for example, may require the knower to construct a variety of collections of knowledge resources known as *concept projections*. By this means, seeing and using the "same" concept in different circumstances requires the use of different (though perhaps overlapping) combinations of knowledge resources.

Because the definite integral lends itself to a variety of different conceptual interpretations, it is a rich area for the study of knowledge flexibility and transfer. A recent series of papers by Jones (2013, 2013/2014, 2015a, 2015b) categorized students' conceptions of the definite integral and argued that different conceptions are more productive than others in the study of physics. The purpose of this paper is twofold. First, using interview data of both beginning and upper-level undergraduate physics students, it will demonstrate the utility of a concept projection as a theoretical construct for knowledge flexibility across levels of expertise. Second, it will expand on Jones' work by examining data revealing both novices' and upper-level physics students' understanding of the definite integral. These findings suggest how concept projections might function to support expert understanding, and point toward opportunities for additional research.

Background: Physics Students' Use of Mathematics

Challenges in transferring mathematics into physics

Both physics and mathematics educators have long observed that even students who have a considerable background in mathematics do not readily use it or apply it in the context of learning physics. Researchers have documented students' challenges in applying ideas from calculus (Christensen & Thompson, 2010; Cui, Rebello, & Bennett, 2006, 2007; Doughty, McLoughlin, & van Kampen, 2014; Nguyen & Rebello, 2011a, 2011b), trigonometry (Ozimek, 2004), and algebra (Torigoe & Gladding, 2011) to concepts and problems in physics. Although some of these researchers have pointed to deficits in students' understanding of mathematics, Yeatts and Hundhausen (1992) and more recently Dray, Edwards, and Manogue (2008) have suggested that students' difficulties result from a "mismatch" or a "gap" between what is taught in mathematics classrooms and what students actually need to use in their study of physics.

Students' understanding and use of the definite integral

A large portion of the research on students' understanding of the definite integral has revealed the limitations in their understanding even after completing a several semesters of calculus. Ferrini-Mundy and Graham (1994) showed the fragility of a student's concepts of integration and other topics in calculus that more recent research continues to find. Most students, it would seem, complete a course in integral calculus with some degree of proficiency in evaluating a definite integral and some knowledge of its applicability to computing the "area under a curve," but students' overall knowledge of procedures, definitions, and underlying concepts are often weak and disconnected (Grundmeier, Hansen, & Sousa, 2006; Mahir, 2009; Rasslan & Tall, 2002).

Evidence suggests the primary interpretation that many students place on the definite integral is an area under a curve (Bezuidenhout & Olivier, 2000; Jones, 2015b), which may limit students' ability to apply integration to other contexts (Sealey, 2006; Jones, 2013, 2015a). Increasingly, researchers have argued that interpreting the integral as a Riemann sum, a sum of (infinitesimal) products, or an accumulation is advantageous to understanding how to use and apply integration to contexts outside of mathematics (e.g., Doughty, McLoughlin, & van Kampen, 2014; Jones 2013, 2015a; Nguyen & Rebello, 2011a, 2011b; Sealey, 2006, 2014; Thompson & Silverman, 2008). These results have emerged in parallel with a growth of research directed toward supporting the development of such understandings in students (Carlson, Smith, & Persson, 2003; Doughty, McLoughlin, & van Kampen, 2014; Engelke & Sealey, 2009; Kouropatov & Dreyfus, 2014).

A recent series of studies by Jones (2013, 2015a, 2015b) has documented a variety of interpretations and understandings that students use to make sense of integration, the definite integral, and its notation. In particular, he found that the most frequent interpretations of the integral used by students could be categorized as *area under a curve*, *antiderivative*, or *multiplicatively based summation* (Jones, 2015b; see also Jones 2013), and of these, the area and antiderivative interpretations were by far the most common. Jones (2015a) further argued that the multiplicatively based summation conception is more productive for sense-making in applied contexts. In this paper, Jones' (2013, 2015a, 2015b) research is used as a basis for critique and development. Further discussion of his work is placed in the analytical sections below.

Theoretical Framework

This study adopts a cognitive, constructivist framework rooted in diSessa's (1993) knowledge- in-pieces epistemology. Wagner (2006) took advantage of the knowledge-in-pieces framework's attention to the context sensitivity of knowledge to use it as a basis for a new understanding of transfer, transfer in pieces. He argued that, contrary to traditional approaches to transfer that presume it takes place due to some abstract nature of knowledge, transfer actually occurs as a learner develops, (re)organizes, and integrates varieties of knowledge resources to accommodate rather than overlook contextual differences.

Wagner (2006) described a concept projection as "a specific combination of knowledge resources and cognitive strategies used by an individual to identify and make use of a concept under particular contextual conditions" (p. 10; see also diSessa & Wagner, 2005; Wagner 2010). From this perspective, recognizing or using a concept (such as a definite integral) in a particular context requires an individual to engage a specific collection of knowledge resources, but the makeup of that collection of resources may vary when the same individual makes use of the same concept in a different contextual situation. The current work takes the applicability of concept projections further by showing that a single individual may use different concept projections in order to "see" or interpret different manifestations of the same concept, in this case, the definite integral, in a single context.

Methods

Student Participants

Students who took part in this study were enrolled in a large public university using a quarter system of eight-week terms. Volunteers included eight beginning students enrolled in an introductory calculus-based physics course focusing primarily on classical mechanics and seven third-year physics majors who had already completed two terms of multivariable (vector) calculus, and at least one additional course in advanced mathematics. For ease of presentation, students will be referred to by a letter and number combination, with beginning students identified as B1- B8 and upper-level students as U1-U7.

Interviews

Students were interviewed individually by the author every other week during the course of an eight-week term, with each interview typically lasting 45-60 minutes. Questions and problems involved conceptual and procedural aspects of integration, differentiation, and other aspects of calculus, some in purely abstract mathematical form, and others in applied contexts. Each segment of an interview typically began with a written question or problem that the student was asked to read aloud, and the student was then asked to respond, thinking aloud as much as possible and explaining his or her thinking as clearly as possible. Further questioning was open-ended and free-flowing. Except in rare circumstances, the interviewer avoided taking on an instructive role. The interviews were audiotaped and videotaped using two cameras and an additional audio recorder.

Data analysis

Analysis of the data took part in stages, using primarily qualitative methods. For the present research, the student's responses and explanations were analyzed and classified according to the type of interpretation of the integral that the student used. In the transfer-in-pieces framework, these categories of interpretation were understood as constituting classes of concept projections. Careful attention was given to students' reasoning strategies and their patterns of use, particular use of language and gesture, the use of intuitive and naive knowledge, and changes and patterns of reasoning across contexts to infer marked characteristics of the different concept projections students used to interpret definite integrals. The goal was not to attempt to specify the entire make-up of any single concept projection, but, by highlighting characteristic knowledge resources that constitute a particular concept projection, to infer differences in concept projections used by an individual under different contextual circumstances, as well as differences in concept projections used by different individuals.

Students' Concept Projections for the Definite Integral

Jones (2013) examined undergraduate students' conceptions of the definite integral and found three principal ways that students interpreted integration and corresponded to normative reasoning. He named these *perimeter and area*, *function matching*, and *adding up pieces* (*multiplicatively based summation*). Jones (2015b) later added one more: *average*. In this paper, I will interpret these categorizations as distinct classes of concept projections. Due to constraints on length, I focus on only two.

The integral as a measure of change

Jones' (2013) *function matching* category for students' reasoning was later identified as an *antiderivative* conceptualization in Jones (2015b). Under his analysis, this interpretation refers to students' perception of the integral as a process of finding an "original function" from which the integrand was derived through differentiation, followed by a process of evaluating the difference of the original function's values at the two endpoints of integration. Although Jones (2015b) permitted the possibility of "a modest layer of meaning" in students' antiderivative conceptualization, he appeared to suggest that this represented a deficient conceptualization:

> However, what is striking is the high prevalence of the anti-derivative conceptions for the definite integral, when anti-derivatives do not actually compose the underlying meaning of the definite integral. It is simply a tool used for calculation purposes à la FTC. (p. 9)

Although it is quite possible that some students hold understandings of the definite integral that support very little sense-making beyond the procedural, the current study found a number of students who made entirely good conceptual and contextual sense of the use of the definite integral to retrieve an "original function."

The RPM Problem
The durability of a car engine is being tested. The engineers run the engine at varying levels of "revolutions per minute" for a period of time. Denote the number of revolutions per minute at time *t* by $R(t)$. Interpret the following:

$$\int_0^{600} R(t)\,dt$$

Figure 1. *The RPM Problem* (adapted from Jones, 2013). The statement of the problem inadvertently omitted units in which time was measured. In all cases, students either made an arbitrary choice of units, or they were told to assume that *t* was measured in minutes.

All of the students in this study were asked to consider the RPM Problem, shown in Figure 1. Beginning student B2 quickly concluded that the integral would determine "how many revolutions there were between time 0 and time 600." He offered the following explanation:

B2: R(t) would be the change in revolutions, in change of time, and if I were to integrate that-. Like that's a form-, it's like a derivative of some function. And if I were to integrate that it would just become a function that was the revolutions rather than the change of revolutions in-, per minute, for example. Revolutions per minute indicates that it's like a ratio of revolutions and minutes. So whatever the integral of this is, it's going to be just an equation that gives you revolutions. And if you were to plug in these values, 600, you would get how many revolutions there were at time 600. And then if you subtracted off revolutions at 0, you would get how many revolutions there were between these bounds.

The student's explanation stands in contrast to Jones' (2015) assertion that "anti-derivatives do not actually compose the underlying meaning of the definite integral." To the contrary, B2 had constructed a more sophisticated antidifferentiation concept projection for the definite integral that allowed him to interpret the antidifferentiation process in a conceptually meaningful manner. His concept projection was composed not only of those procedural resources identified earlier, but also of additional interpretive resources. In addition to understanding differentiation and integration as reversible procedures, he invoked resources that enabled him to reverse the interpretation of the derivative of a quantity as the rate of change of that quantity. In this way, he could conclude that integrating turned a rate of change of something ("the change of revolutions in-, per minute") into a function that gave an amount of that something. Further resources allowed him to interpret the substitution and subtraction procedure as a means of finding how much that something changed over the interval of integration.

In a pure math context, or faced with a need simply to evaluate an integral, B2 would probably not need to invoke all of the resources at his disposal. But in a contextually meaningful situation, he gathered a rich collection of interpretive resources to construct a concept projection that gave meaning to the function matching procedures. B2 was not alone. Among all the participants of this study, three beginning students and two upper-level students all demonstrated an ability to use an equally meaningful concept projection for the antidifferentiation process. Although these concept projections are elaborations on *function matching*, they may deserve to comprise a class of their own, which I call *integration as change*.

The example of B2 offers a nice opportunity to highlight the context-sensitivity of concept projections. Although B2 (and others) could be observed using an *integration as change* concept projection in some circumstances, both B2 and another student could not use it to interpret integrals whose integrand was not known to them as a derivative or a rate of change. When asked to consider an integral whose integrand was a position function, B2 concluded that the integrand made no sense because there had to be "a change in something else." He concluded, "I don't really know if that correlates to real life." Similarly, another beginning student who showed herself able to use an *integration as change* concept projection in some circumstances denied that one could integrate a force function, because she believed one could integrate only "a function that gives a rate," and she did not perceive a force as a rate. Such examples demonstrate the delicate relationship between mathematical knowledge and contextual understanding.

One should not conclude, however, that an *integral as change* concept projection is somehow poor or deficient. It serves a perfectly fine purpose in appropriate contexts, and it demonstrates that students who may appear to be using a purely procedural function matching interpretation of the definite integral may well be trying to engage in meaningful sense-making.

The integral as a weighted average

Jones (2015b) added the notion of an integral as an *average* to his earlier list of categories for integral interpretations. In this conceptualization, the integral is interpreted "as a process that takes a non-uniform function and 'smooths' it out over the domain to make it as though it were uniform" (p. 11). The uniform function represents the average value of the integrand over the interval of integration, and the value of the integral is the area under the graph of the average value over the same interval. Jones's data did not permit him to investigate the basis for this conception, however, and he initially framed it as potentially in conflict with "the *underlying meaning* of the integral," and he suggested that it might be the result of "interesting circular reasoning" (p. 12, emphasis original).

One of the upper-level students in the present study, U2, also demonstrated an ability to conceptualize the definite integral using an averaging process, and, over the course of several interviews, he offered substantial explanations for his understanding that are not rooted in circular reasoning. I argue that he developed a *weighted average* concept projection for the definite integral that is entirely sensible, based on a variation of a *sum of products* concept projection, and, perhaps surprisingly, powerful in its ability to enable him to make intuitive sense of an integral that is otherwise quite difficult to interpret. A thorough analysis of U2's understanding of integration requires lengthy and detailed investigations of his thinking over several problems and several interviews. What I provide here will necessarily be an insufficient summary of that analysis.

In U2's discussion of the definite integral in the abstract (ie, apart from a specific problem or application), he demonstrated an ability to interpret the integral as sum of products. He resisted the notion of the differential as an "infinitesimal," however, and so he preferred to avoid the language of products, concluding instead that "you're doing a two dimension sort of summing that "coupled" the integrand with the differential in a way that measures how "present" the values of the integrand are over the interval of integration. Several times throughout the interviews, he referred to the "presence" of the integrand, usually using the gesture of air-quotes to highlight his use of the word. In further discussion, U2 proposed the following:

> U2: I wonder if it could be explained with like a weighted average kind of, where you are weighting each number in the range by its, kind of like, "presence" [indicates air quotes] in the range, where its-, each number has an infinitely small presence in the range of this like weighted average. But we can still compute it, because integrals allow us to do that.

U2 used the term *range* to refer to all the possible values of the integrand over the interval of integration. When I asked him what, specifically, was doing the weighting, he replied, "the weight of each value is the width of that value," and referred to a graph of a Riemann sum area approximation, clearly indicating that the "width" referred to the rectangles, or the role of the differential in the integral. He superimposed a horizontal line over an existing graph of a non-constant function, calling it the average value of the function, and concluded, "if you just find the area of this shape [shades the rectangular part], you would have the area of the original shape." This is precisely what Jones (2015b) reported observing with some of his students.

From a mathematical perspective, the overall argument that U2 made is entirely sensible, if accompanied by some clarifying detail. Technically, for example, finding the (weighted) average of the integrand requires weights equivalent to the width of the subintervals divided by the width of the interval of integration, but since the actual average is never found, U2 never approached the question at that level of detail. The point is that U2 constructed a concept projection for the integral that permits a way to imagine how the value of the integral is found (not simply, as Jones suggested, how to interpret its answer). In a follow-up discussion, U2 also indicated that he was aware that the actual value of the average value of the function is never actually found or needed, rather, "It's kind of like a conceptual tool…, so there is no average that ever happens."

U2's developed notions of interpreting the integral as measuring the "presence" of values of the integrand were pervasive throughout his interviews. He spontaneously explained at one point, that if one were to integrate the constant function $f(x) = 7$, the result could be interpreted as "the sevenness that's done between these bounds." He reaffirmed it: "Its seven-ness. That makes sense to me."

This way of looking at the integral is not simply novel; it actually has some power behind it. I asked all students in this study to consider the meaning of an integral of a position function over an interval of time. In practice, this integral has no common interpretation, and I was primarily interested in whether or not students could deduce its units. Nonetheless, U2 was the only student who was able to give a rather insightful interpretation of the integral:

> U2: So the units of your integral are going to be distance times time, since you're integrating over time. And so [...], so I guess, yeah, my brain can't interpret the physical-. I'm trying to think of like a real world problem that would do something like this, and, I don't know, like, "awayness," [uses air quotes] like, you wanted to figure out how far a particle was from a location where both distance and time are important.

U2's interpretation of the integral as a measure of "awayness" clearly comes from his weighted average concept projection. He paralleled his language of "sevenness" used above, and he reintroduced the air quotes he used in the past when he spoke of the integral as a measure of the "presence" of the integrand. His concept projection allowed him to offer the only conceptual interpretation of the integral of distance with respect to time suggested in this study. To my eyes, it is a lovely interpretation, capturing, as he noted, the sense that, when one is away, "both distance and time are important."

Discussion

Concept projections offer a way to consider how a variety of different meanings can be (are!) constructed to interpret and make sense of integration. Which meaning is most advantageous or useful to any individual in any particular circumstance can and will vary. In many cases, for both students and experts, interpreting an integral using an *integral as change* concept projection is entirely sufficient, without any appeal to Riemann sums. It is a simpler, perhaps more direct, way of making sense of integration through antidifferentiation, it holds up mathematically, and it appeals to the meaning of the antiderivative. Furthermore, using a *weighted average* concept projection is also a legitimate way to see meaning in the integration process, and one that carries its own interpretive advantages. It is true that nowhere in the process of integration does one actually find or use the average value of the integrand. It is, as U2 indicated, a conceptual tool. But it is equally true that nowhere in the integration process carried out through antidifferentiation does one actually find rectangles, infinitesimals, or sums. Riemann sum interpretations are also conceptual tools. There is no single "meaning" of the definite integral.

There are ongoing efforts being made to expand opportunities for students to learn more conceptually sophisticated interpretations of the definite integral, particularly Riemann sum-based interpretations. I believe that thinking of the educational task at hand as supporting students in developing a variety of concept projections for the integral can be helpful in developing and measuring the success of these efforts. The repeated message that comes through research based on the knowledge-in-pieces and transfer-in-pieces frameworks is that contextual differences that experts have learned to think about as irrelevant, only appear irrelevant after engaging with them enough to construct the cognitive resources required to accommodate them.

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References

Bezuidenhout, J., & Olivier, A. (2000). Students' conceptions of the integral. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education*, *2*, 73-80.

Carlson, M. P., Smith, N., & Persson, J. (2003). Developing and Connecting Calculus Students' Notions of Rate-of Change and Accumulation: The Fundamental Theorem of Calculus. *International Group for the Psychology of Mathematics Education*, *2*.

Christensen, W. M. & Thompson, J. R. (2010). Investigating student understanding of physics concepts and the underlying calculus concepts in thermodynamics. *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education*. Raleigh, NC. Downloaded 8 June 2015 from http://sigmaa.maa.org/rume/crume2010/Archive/Thompson%20&%20Christensen.pdf

Cui, L., Rebello, N. S., & Bennett, A. G. (2006). College students' transfer from calculus to physics. *Proceedings of the 2005 Physics Education Research Conference*, *Vol. 818*, pp. 37-40.

Cui, L., Rebello, N. S., & Bennett, A. G. (2007). Assessing college students’ transfer of learning from calculus to physics using non-traditional problems. *Proceedings of the NARST 2007 Annual Meeting*. New Orleans, LA.

diSessa, A. A. (1993). Toward an epistemology of physics. *Cognition and Instruction*, *10*, 105–225.

diSessa, A. A. & Wagner, J. F. (2005). What coordination has to say about transfer. In J. P. Mestre (Ed.), *Transfer of learning from a modern multidisciplinary perspective* (pp. 121-154). Greenwich, CT: Information Age Publishing.

Doughty, L., McLoughlin, E., & van Kampen, P. (2014). What integration cues, and what cues integration in intermediate electromagnetism. *American Journal of Physics, 82,* 1093-1103.

Dray, T., Edwards, B., & Manogue, C. A. (2008). Bridging the gap between mathematics and physics. Retrieved 8 June 2015, from http://tsg.icme11.org/document/get/659

Engelke, N., & Sealey, V. (2009). The great gorilla jump: A Riemann sum investigation. *Proceedings of the 12th Annual Conference on Research in Undergraduate Mathematics Education*. Raleigh, NC.

Ferrini-Mundy, J., & Graham, K. (1994). Research in calculus learning: Understanding of limits, derivatives, and integrals. *MAA NOTES*, 31-46.

Grundmeier, T. A., Hansen, J., & Sousa, E. (2006). An exploration of definition and procedural fluency in integral calculus. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *16*(2), 178-191.

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior, 32*(2), 122-141.

Jones, S. R. (2013/14). Adding it all up: Reconceiving the introduction of the integral. *Mathematics Teacher, 107*(5), 372-377.

Jones, S. R. (2015a). Areas, anti-derivatives, and adding up pieces: Integrals in pure mathematics and applied contexts. *The Journal of Mathematical Behavior, 38*(1), 9-28.

Jones, S. R. (2015b). The prevalence of area-under-a-curve and anti-derivative conceptions over Riemann sum-based conceptions in students' explanations of definite integrals. *International Journal of Mathematics Education in Science and Technology, 46*(5), 721-736.

Kouropatov, A., & Dreyfus, T. (2014). Learning the integral concept by constructing knowledge about accumulation. *ZDM*, *46*(4), 533-548.

Mahir, N. (2009). Conceptual and procedural performance of undergraduate students in integration. *International Journal of Mathematical Education in Science and Technology*, *40*(2), 201-211.

Nguyen, D. H., & Rebello, N. S. (2011a). Students' understanding and application of the area under the curve concept in physics problems. *Physical Review Special Topics-Physics Education Research*, *7*(1), 010112.

Nguyen, D. H., & Rebello, N. S. (2011b). Students' difficulties with integration in electricity. *Physical Review Special Topics-Physics Education Research*, *7*(1), 010113.

Ozimek, D. J. (2004). Student Learning, Retention, and Transfer from Trigonometry to Physics. Unpublished Master's thesis, Kansas State University, Manhattan, KS.

Rasslan, S. & Tall, D. (2002). Definitions and images for the definite integral concept. In A. D. Cockburn & E. Nardi (Eds.). *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp 89-96). Norwich, UK.

Sealey, V. (2006). Definite integrals, Riemann sums, and area under a curve: What is necessary and sufficient. *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, p. 46).

Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior*, *33*, 230-245.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. *Making the connection: Research and teaching in undergraduate mathematics*, *73*, 43-52.

Torigoe, E. T., & Gladding, G. E. (2011). Connecting symbolic difficulties with failure in physics. *American Journal of Physics*, *79*(1), 133-140.

Wagner, J. F. (2006). Transfer in pieces. *Cognition and Instruction, 24*(1), 1-71.

Wagner, J. F. (2010). A transfer-in-pieces consideration of the perception of structure in the transfer of learning. *Journal of the Learning Sciences, 19*(4), 443-479.

Yeatts, F. R., & Hundhausen, J. R. (1992). Calculus and physics: Challenges at the interface. *American Journal of Physics*, *60*(8), 716-721.