The graphical representation of an optimizing function

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Optimization problems in first semester calculus present many challenges for students. In particular, students are required to draw on previously learned content and integrate it with new calculus concepts and techniques. While this can be done correctly without considering the graphical representation of such an optimizing function, we argue that consistently considering the graphical representation provides the students with tools for better understanding and developing their optimization problem-solving process. We examine seven students' concept images of the optimizing function, specifically focusing on the graphical representation, and consider how this influences their problem-solving activities.

Key words: Calculus, Optimization, Function, Graphs

Word problems are notoriously challenging for students - not a surprise to anyone who has ever been a student or an instructor of mathematics. Slightly obscuring the mathematics involved by describing a situation using everyday "nonmathematical" language adds an extra level of difficulty. In first semester calculus, students have many opportunities to solve word problems. Here we examine optimization problems, which require students to read a short description of a scenario in which a quantity needs to be maximized or minimized. This quantity may be an area, a volume, a cost, a distance, an amount of material, or a production output. To solve the problem, the student must construct a function for this quantity (we call this the *optimizing function*) and then use calculus techniques to find the absolute maximum or absolute minimum of the function in the appropriate domain. White and Mitchelmore (1996) found that students are much more likely to be able to find a desired maximum or minimum when the function is given explicitly in the problem than they are when the problem is stated in the form of a word problem and the students must first construct the appropriate function. For this reason, we are interested in studying how students construct the optimizing function and how this influences their understanding of the rest of the problem.

Our research is guided by the following two research questions: 1) What facets of learners' concept images influence their construction of the optimizing function when solving calculus optimization problems? and 2) How can this shed light on teaching interventions that could support conceptual development of optimization?

LaRue and Engelke Infante (2015) identified six key mathematical concepts that play a role in students' understanding of optimization – specifically their construction of the optimizing function. These six mathematical concepts are: variables, function notation, function composition, properties of rectangles and the relationships between them, the role of the optimizing function, and the graphical representation of the optimizing function. Here, we examine in greater depth the students' concept images of the optimizing function, focusing on those aspects related to the ability to transition between the graphical and algebraic representations of the optimizing function throughout the problem-solving process. We observed that students' inclination to consider the graphical representation of the optimizing function was directly related to their ability to explain the reasoning for their work and to solve challenging optimization problems.

Literature Review

Graphical interpretations of functions can convey information about the functions in a single image. The graph of a function can benefit students because it allows them to consider the overall behavior of the function, rather than focusing on individual elements. Students, however, are often reluctant to consider the graphical representation of a function, and when they do, they frequently have trouble interpreting the information correctly (Eisenberg, 1992). Sfard (1992) noted, “Graphs provide another way of thinking about functions, but there is almost no connection between a graph and the underlying formula” (p. 75). Knuth (2000) reported, “three fourths of the students chose an algebraic approach as their primary solution method, even in situations in which a graphical approach seemed easier and more efficient than the algebraic approach” (p. 504). Even (1998) found that students have trouble solving problems that require them to move seamlessly between different representations of functions.

Arcavi (2008) noted that analytic techniques are frequently “devoid of meaning” for students and suggests having students examine the graph of a function *prior* to using analytic techniques to determine information about the function (p. 9). He suggested using a dynamic graphing tool that allows the students to watch the function being drawn and to see the relevant information about the function at various points on the function. He argued, “a dynamical graphical model highlights aspects of the situation that were not as salient had we investigated it alone or even by modeling it symbolically” and gave the example of using dynamic graphs to examine the relationship between the perimeter of a rectangle and the length of one its sides and the relationship between the perimeter of a rectangle and the length of its diagonals (p. 5).

Theoretical Perspective

Tall and Vinner (1981) define a learner’s concept image as ‘all the cognitive structure in the individual’s mind that is associated with the given concept’ (p. 1). Because the concept image exists in the mind of the student, and we know that students do not always have correct understandings of mathematical concepts, this cognitive structure may be incomplete, incorrect, or logically inconsistent. When a need arises, the parts of the concept image that are directly related to the need are called upon, but the rest of the concept image remains dormant, ready to be accessed if needed, but not until then. This means students may have conflicting information in their concept images without realizing it, and unless the two logically inconsistent parts of the concept image are evoked simultaneously, the student may never realize something is wrong.

Carlson and Bloom’s (2005) problem-solving framework allows us to describe students’ activity as they solve optimization problems. The framework is divided into four main phases: orienting, planning, executing, and checking. In the orienting phase, the student deciphers the problem and assembles the tools he or she thinks may be required. In the planning phase, the student uses conceptual knowledge to determine an appropriate course of action, which is then implemented during the executing phase. Finally, during the checking phase, the problem solver goes back to the original problem to see if the answer makes sense.

During the orienting phase, students will likely focus on algebraic aspects of the problem as they assemble useful formulas and equations. In the planning phase, we would expect students to work to construct an appropriate optimizing function. It is during this phase that we would like to see them consider the graphical representation of the function as a tool for quickly recognizing the algebraic techniques they need to employ to solve the problem. Considering the graphical representation should help them determine that the key next steps are to differentiate, find critical points, and use either concavity or the increasing and decreasing nature of the graph to verify that they have solved the problem. Much of the executing phase is computational in nature and will focus on the algebraic representation of the function. Finally, during the checking phase, the graphical representation of the optimizing function affords students the opportunity to verify that their answer makes sense.

Methods

Data was collected through a series of semi-structured interviews with first semester calculus students at a large state university in the United States. Interviews were conducted during the summer and fall semesters of 2014. Four students (Franz, Sam, Tracy and Lars) were interviewed during the summer of 2014 and three students (Ashod, Brandi, and Cy) were interviewed during the fall of 2014. In both cases, the students were interviewed after their exam covering optimization and just before their final exam for the class. The students were selected on a volunteer basis and self-reported average to strong mathematical backgrounds. All interviews were video recorded and transcribed for analysis. We used open and axial coding to isolate and further analyze portions of the interviews associated with the graphical interpretation of the optimizing function.

The students were asked to solve the following optimization problem, which is standard in most first semester calculus classes: *A rectangular garden of area 200 ft^2 is to be fenced off against rabbits. Find the dimensions that will require the least amount of fencing if a barn already protects one side of the garden.* We refer to this problem as the **garden problem**. After solving this problem, the students were asked questions about the connection between the area and the perimeter of rectangles and then were asked to solve an additional optimization problem involving the volume and surface area of a 3D object. The students interviewed in the summer of 2014 were asked to examine a graph of the optimizing function associated with the second optimization problem, while the students in the fall of 2014 were asked to examine a graph of the optimizing function associated with the garden problem and with the second problem. In both cases, the graphs consisted of two unlabeled axes and a rough sketch of the function. The students were asked to mark and label important information on the graphs.

Results

The seven students interviewed had varying levels of success when asked to transition from the algebraic expressions of the optimization problem to the graphical representation. We have grouped them based on the strength of their graphical connections, and we discuss these below.

No Graphical Connections: Franz and Ashod

Franz and Ashod did not have well-developed concept images of the optimizing function. Ashod constructed his optimizing function with the motivation of finding something to differentiate and set the derivative equal to zero because, “when it equals zero, that’s when you know you have the least amount.” His only rationale for deciding how to construct such a function, however, was, “whatever they give you, use the other equation.” When he was given a graph of the function he had constructed and was asked to explain how the answer to the problem related to the graph, he said they were not related, and he was unable to correctly mark the place where his answer belonged on the graph (see Figure 1). He knew he had been using the derivative to solve for the answer, and since "the graph of the derivative looks completely different from the graph of the original equation," he did not think his answer related to the original graph at all. He was able to explain that the first derivative always tells you whether the graph is "positive or negative or increases or decreases," but even though this was part of his concept image for the first derivative of a function, he did not relate this information back to the graph of the optimizing function. Thus, Ashod's concept image for the optimizing function was developed just enough to allow him to be able to solve the problem and get an answer, but not enough for him to be able to move from the algebraic interpretation to the graphical interpretation.

Franz was unable to make any connections to the graphical representation of a function when solving the garden problem. Initially, he tried to set the optimizing function equal to zero and solve for *x*, but when he got a negative answer, he realized that wouldn't work. His next attempt was correct (setting the derivative equal to zero), but when he was asked why he was doing that, he said, "it's a standard thing that we do," and, "I have no idea. I just know that is the minimum." The interviewer repeatedly encouraged him to make connections to the graph, but with no success. When he was asked to label the graph of the optimizing function, he decided where to put his answer based on where he thought that number would generally be located on a number line (see Figure 1), completely disregarding the shape of the graph. His responses indicate that his concept image for the optimizing function contained little more than some basic facts about what to do with it, without any links to the graphical representation of the function or the context of the problem.

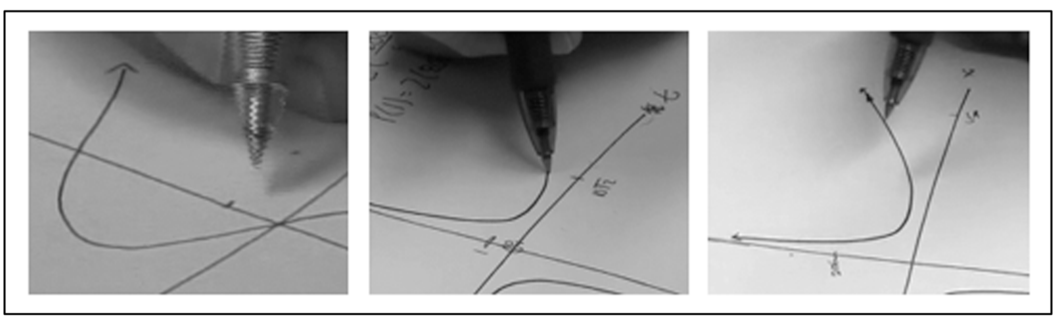

Figure 1. Franz, Brandi, and Ashod mark the location on the graph where they believe the critical number belongs on the axis. Note that all three students placed the mark somewhere other than below the obvious maximum or minimum.

Limited Graphical Connections: Cy and Lars

Cy and Lars were able to use the language of graphs to discuss the algebraic work they had done, but they had trouble making some connections initially. Cy knew there was a connection between the graph of the optimizing function and the algebraic expressions he was working with. Early in his explanation, he stated that the first derivate indicates where "the slope equals zero" and the second derivative indicates the concavity, signifying whether there is a maximum or minimum. However, Cy had a lot of difficulty interpreting the graph when it was first presented to him. Like Ashod, he was confused because he had used the derivative to solve for his answer, and was thus did not understand how his answer could have something to do with the graph of the original function.

Cy recognized that making the transition from the algebraic representation to the graphical representation made sense and was surprised that he was having trouble. He said, “I don’t know why this is so hard. This seems like something very easy.” Later in the interview, he said, “It’s weird to like make the jump from numbers into a graph sometimes. In some situations, the numbers are really just a stand in for the work I’m doing in my head with graphs, but in this situation, it’s more, the numbers are all I really ever thought about with this.” This is especially interesting, because he used the language of graphs when he was explaining his work, but was still unable to translate that to an actual graph. Eventually, with some prompting from the interviewer, he was able to make sense of the graph and connect it to the problem.

Lars brought up graphs without being prompted. He had trouble constructing his optimizing function, but explained that all of the information could be put on a graph. Unfortunately, even though he knew that there was a graphical component to the optimization problem, he did not know what function corresponded to the graph. He could not label the axes correctly or give any sensible information about what information could be obtained from the graph. At one point he thought the two axes corresponded to the two sides of the rectangular garden, and at another point he thought the two axes corresponded to one side of the garden and the area of the garden. Eventually, after a lot of intervention from the interviewer, Lars was able to construct an appropriate optimizing function and correctly relate it to the graphical representation. So even though he began with the recognition that he could use a graph to figure out how to solve the problem, he did not know what the graph should represent. Once he realized that the graph should represent the amount of material needed for the fence, he was able to complete the problem.

Strong Graphical Connections: Tracy, Sam and Brandi

Tracy began the problem by trying to recall how she had done similar problems, but quickly became confused and unsure how to proceed. She knew that she was trying to find the minimum of a function and that she could find that by taking the derivative, saying, “the derivative would just be the rate of change of the graph, so the best I can figure out of that is finding the critical values in the derivative would help you find the minimums because you look at areas of increase and decreases.” When she was asked why she wanted to set the derivative equal to zero, she said, “When the derivative is zero? Doesn’t it just have a horizontal tangent line which means it has to have that shape, the parabola shape?”

Unfortunately, even though she knew this, she did not know what function she should be differentiating. After a lot of intervention from the interviewer, she was able to recognize that for the garden problem, she was trying to construct a function representing the amount of material used to construct the fence. Once she figured that out, she was able to solve the rest of the problem easily, because she had such a strong understanding of the connections to the graphical representation of the optimizing function.

Sam very quickly set up and solved the garden problem with little difficulty. He explained, “we take the derivative of that function in order to find the points, uh, where slope equals zero in order to tell us where it stops increasing, decreasing, and that’ll tell us where the minimum or maximum values are.” However, as the interview progressed, he realized that he did not understand how the optimizing function, and particularly the *graph* of the optimizing function, related to the problem. He said, “Like how this fence has a minimum value that relates on a graph that’s a function of a different function.” He recognized that his initial equation represented the amount of fence of the garden, but once he eliminated one of the variables and constructed a single-variable function, he could not relate this "new" function to the amount of fencing. Thus, he knew graphical properties about functions in general, but he did not understand how the graph was related to his original algebraic expression.

When Brandi was asked to explain her work, one of her first responses was, "Cause like the amount of feet that could be used, if you think about it on the graph." She immediately made the connection to the graphical representation of the optimizing function, indicating that it is a well-developed part of her concept image. She was able to talk about this clearly and comfortably about the relationship between the graphical representation and the algebraic representation as she explained her thought processes, yet when she was presented with the actual graph, she had trouble marking the correct place for her answer, $10\sqrt{2}$. She placed it where she thought $10\sqrt{2}$ would fall on the axis (see Figure 1), not directly below what was clearly the minimum of the function. On a theoretical level, she appeared to understand the connection, but when she had to apply what she knew to an actual graph, she still had some difficulty.

Tracy, Sam, and Brandi had more developed understandings of the connections than the other students, and they moved more flexibly between the algebraic and graphical representations. All three, however, still had some difficulties with the two representations.

Discussion

The students in this study were not naturally inclined to consider the graphical representation of the optimizing function when solving optimization problems. When they did consider the graphical representation, most did so incorrectly or with an incomplete understanding of how it was related to the algebraic representation and the work associated with it.

In the planning phase of Carlson and Bloom's (2005) problem-solving framework, the student determines an appropriate course of action for solving the problem. Ideally, for the garden problem, students would consider that they need to construct a function representing the amount of fencing required for an area of $200ft^2$. They would then consider the graphical representation of this function and recognize that since they need to find the minimum, they should expect their function to be concave up at the value they find.

During this phase, some students recalled similar problems and simply attempted to duplicate a familiar solution path. Franz, Ashod, Lars, and Tracy all began this way. When this didn't work for Lars and Tracy, they were (with some encouragement) able to fall back on their knowledge about the graphical representation of the optimizing function to figure out how to move forward. Franz and Ashod were able to solve the first problem without intervention, but both had trouble solving the second more difficult optimization problem. They did not have any understanding of the graphical representation of the function to fall back on and were unable to determine how to move forward. Tracy did have this understanding, but she had so much trouble with the 3-D aspect of the second problem that she was unable to solve the problem without a lot of help. However, once she reached an answer, she was able to clearly explain what her answer meant in the context of the problem, suggesting that if she had a stronger concept image of surface area and volume, she would have been able to solve the problem on her own. Lars was able to solve the second optimization problem with ease, because once he had reasoned through the connection to the graphical representation once, he was able to draw upon this to make sense of the next problem.

The other three students, Brandi, Cy, and Sam, had some difficulties, but all began the first problem with well-developed concept images for the optimizing function that included at least some understanding of the graphical representation of the function. They began solving the problem by referencing this graphical connection and were able to give explanations other than "I don't know," or "because this is how my teacher did it" when they were asked why they were beginning the problem that way. Additionally, when they attempted to solve the more difficult second optimization problem, they were all successful. For these students, a well-developed concept image for the optimizing function, including at least some understanding of the connection to the graphical representation, led to more success in solving and understanding optimization problems.

Thus, we see that the graphical representation of the optimizing function has an important role to play in helping students develop their understanding of optimization in general. Because existing literature and our current research tell us that most students generally are not likely to move fluidly between the algebraic and graphical representations, we must work to find ways to encourage students to make these connections.

Conclusion

We found that even when students were able to accurately describe the connection between the algebraic and graphical representation of the optimizing function, they often had more difficulty when they were asked to put this information to use when dealing with an actual sketch of the graph. We suggest asking the following questions when teaching and/or assessing students on optimization problems.

1. Identify your optimizing function. What does it represent? How do you know it is a function?
2. What is the realistic domain of your optimizing function? What is the realistic range?
3. Draw a rough sketch of your optimizing function. Label the axes appropriately.
4. Consider an ordered pair *(a,b)* on the function. In the context of the problem, what does *a* represent? What does *b* represent? In the context of the problem, what is the relationship between *a* and *b*?
5. Mark the point in the domain of the function that corresponds to the answer you hope to find (or have already found) using algebraic techniques.

These questions are designed to encourage the students to think about and make the connections between the different representations of the optimizing function and to help them further develop their concept image of the optimizing function. Making these connections will help the students set up and solve these problems, particularly during the planning and checking phases of the problem-solving process.

In our study, the graphical representation of the optimizing function was only a portion of the interview protocol, but it has emerged as a significant theme in our research. We believe that there is room for a more targeted, small-scale research project focused on examining the role that the graphical representation of the optimizing function plays in students' work with optimization problems. The above questions could be a good starting point for such a project.

References

Arcavi, A. (2008). Modeling with Graphical Representations. *For the Learning of Mathematics, 28*(2), 2-10.

Carlson, M., & Bloom, I. (2005). The Cyclic Nature of Problem Solving: An Emergent Multidimensional Problem-Solving Framework. *Educational Studies in Mathematics, 58*(1), 45-75.

Eisenberg, T. (1992). On the development of a sense for functions *The concept of function: Aspects of epistemology and pedagogy* (Vol. 25, pp. 153-174).

Even, R. (1998). Factors involved in linking representations of functions. *The Journal of Mathematical Behavior, 17*(1), 105-121.

Knuth, E. J. (2000). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education*, 500-507.

Sfard, A. (1992). Operational Origins of Mathematics Objects and the Quandry of Reification - The Case of Function. *The Concept of Function, Aspects of Epistemology and Pedagogy, MAA Notes, 25*, 59-84.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

White, P., & Mitchelmore, M. (1996). Conceptual Knowledge in Introductory Calculus. *Journal for Research in Mathematics Education, 27*(1), 79-95.