Limitations of a "chunky" meaning for slope

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This paper will investigate the question "What mathematical meanings do high school mathematics teachers hold for slope and rate?" It will also investigate to what extent these meanings for slope and rate are multiplicative, that is built on an image of quotient as a measure of relative size. A multiplicative meaning for rate of change is powerful because it allows the teacher to better differentiate between additive and multiplicative situations. The data comes from the administration of the diagnostic instrument named "Meanings for Mathematics Teaching Secondary Math" (MMTsm).

Key words: Secondary Teacher Preparation, Slope, Rate, Diagnostic Instrument

Coper-Gencturk (2015) followed 21 K-8 teachers for three years to determine how their mathematical knowledge and teaching changed over time. She found that the improvement in teachers' mathematical knowledge as a result of the master's program and their overall level of mathematical knowledge played significant roles in "indicating the extent to which teachers were successful in constructing meanings for mathematical rules and articulating what mathematical ideas students were supposed to learn" (Coper-Gencturk, 2015, p. 314). Teachers with lower content knowledge for teaching made superficial changes to their instruction such as putting students in groups to discuss procedures, or adding real-world or hands-on activities that were not clearly connected to the mathematical ideas being taught (Coper-Gencturk, 2015). It is important to understand and address mathematical weaknesses of teachers to help them implement meaningful changes in their classrooms.

Studies that employed time intensive methods such as interviews to study small samples (less than 10) of teachers described teachers who conveyed computational or additive meanings for slope (Coe, 2007; Stump, 1999). Project Aspire developed the instrument named *Mathematical Meanings for Teaching Secondary Mathematics (MMTsm)* to help professional developers and researchers more quickly and meaningfully diagnose teachers' mathematical thinking. This study builds on prior researchers' understandings of teachers' meanings for slope and rate, by investigating the following questions in a much larger sample of teachers using items related to slope and rate of change from the *MMTsm*.

1. What meanings for slope and rate might teachers' responses convey to their students?
2. To what extent do these meanings build on an image of quotient as a measure of relative size?

Theoretical Perspective

Thompson's (2013, 2015) work on meaning is the theoretical foundation of Project Aspire. Thompson defined "meaning" in the context of earlier research on the development of children's mathematical schemes. Harel and Thompson used the Piagetian notion of scheme to define a *stable* meaning as the "the space of implications that results from having assimilated to a scheme. *The scheme is the meaning*" (Thompson, Carlson, Byerley, & Hatfield, 2014, p. 13). Glasersfeld (1995) identified the three parts of schemes as follows:

1. Recognition of a certain situation.
2. A specific activity associated with that situation.
3. The expectation that the activity produces a certain previously experienced result (p. 65).

A person's meaning for a mathematical idea includes both what comes to mind when they encounter an idea and what is immediately implied by whatever comes to mind—what might come to mind easily next.

Literature Review

The explanations of constructs will use examples from interviews with teachers conducted in prior qualitative research. We will explain one non-multiplicative "chunky" way of thinking about slope and the limitations of this way of thinking. According to Castillo-Garsow (2010; 2012) a "chunky" way of thinking about quantities changing entails imagining completed chunk, that is an unit chunk. Thus, an individual using a "chunky" way of thinking is likely to imagine only changes in chunks instead of continuous change. Stump (2001) interviewed pre-service teachers named Joe, Tracie and Natalie and observed their teaching as part of a study on pre-service teachers' understandings of slope and how they expressed their meanings in the classroom. Joe planned and taught lessons on slope after discussions in a methods course designed to help teachers develop stronger meanings for slope. "Joe eventually defined slope as 'vertical change/horizontal change,' and presented a graph of the line passing through the points (0,0) and (3,2). He emphasized that the slope as a fraction, 2/3, up 2, over 3" (Stump, 2001, p. 216). One student in Joe's class "was having difficulty understanding how the two fractions 5/-6 and -5/6 could both represent the same slope. Although at the time Joe struggled in vain to help her understand, he later described her difficulty with the following insight: 'They think you are describing a movement as opposed to you describing a number, a measurement'" (Stump, 2001, p. 216). Although Joe's personal meanings were sufficient to allow him to see '5/-6' and '-5/6' as the same slope, the meaning for slope he conveyed to the student (namely, slope tells us how to go up and over) limited the student's ability to use slope productively. Further, the meaning for slope Joe conveyed to this student was strongly connected to the conventional Cartesian coordinate system and the act of moving over and up in chunks of 2 and 3. His meaning for slope could not be applied to polar coordinate systems or real world situations where two quantities change together, but do not move horizontally and vertically.

Joe conveyed a chunky, non-multiplicative meaning for slope because he did not say for any size change in x the change in y is 2/3 as large. Other teachers also did not strongly connect the idea of slope to the notion of a quotient as a measure of the relative size of the change in x and the change in y. Coe (2007) asked Peggy "why do we use division to calculate slope?" and she replied that she didn't know because "she never really thought of it as the division operation" (p. 207). Even though Peggy realized that there is a division symbol in the formula for slope she seemed not to have questioned how it related to her meanings for division.

Some teachers' tendency to avoid using a multiplicative meaning for quotient in explanations of slope may be because their meanings for quotient are weak. McDiarmid and Wilson (1989) gave a written instrument to 55 alternatively certified secondary teachers with mathematics degrees. He presented them with four story problems and asked them to choose which story problem could be solved by dividing by ½. Only 33% were able to identify quantitative situation that involved division by a fraction. In interviews by McDiarmid and Wilson (1989) some alternate route secondary teachers could see no real world application for division by fractions.

Ball (1989) asked prospective teachers "to develop a representation—a story, a model, a picture, a real-world situation—of the division statement $1\frac{3}{4} \div \frac{1}{2}$" (p. 21). Five out of 9 prospective secondary teachers and 0 out of 9 elementary teachers were able to generate an appropriate representation (p. 22). Byerley and Hatfield (2013) asked 17 pre-service secondary teachers to draw a picture representing a particular division problem. Six out of 17 were able to represent the relative size of 7.86 and .39 in an image to explain the meaning of a quotient (Byerley & Hatfield, 2013). Without an image of quotient as a measure of relative size, it is hard to build a meaning for slope as a measure of the relative size of the change in x and the change in y.

Item Development

The motivation for Project Aspire was to design items and scoring rubrics that allow researchers and teacher educators to categorize teachers' meanings with a written diagnostic instrument. Thompson (2015) summarized the process of creating items and rubrics for the *MMTsm*:

> (1) Create a draft item, interview teachers (in-service and pre-service) using the draft item. A panel of four mathematicians and six mathematics educators also reviewed draft items at multiple stages of item development. In interviews, we looked for whether teachers interpret the item as being about what we intended. We also looked for whether the item elicits the genre of responses we hoped (e.g., we do not want teachers to think that we simply want them to produce an answer as if to a routine question); (2) Revise the item; interview again if the revision is significant; (3) Administer the collection of items to a large sample of teachers. Analyze teachers' responses in terms of the meanings and ways of thinking they reveal; (4) Retire unusable items; (5) Interview teachers regarding responses that are ambiguous with regard to meaning in cases where it is important to settle the ambiguity; (6) Revise remaining items according to what we learned from teachers' responses, being always alert to opportunities to make multiple-choice options that teachers are likely to find appealing according to the meaning they hold; (7) Administer the set of revised items to a large sample of teachers.

We designed the item in *Figure 1* to reveal teachers' meanings for slope in the context of teaching. The inspiration for the name of the item came from Coe (2007) and his observations that the three teachers he interviewed did not connect the idea of slope with a measurement meaning of division. We designed Part B to prompt teachers to reflect on the relationship between any size change in x and the associated change in y because we anticipated many teachers would give responses to Part A that were similar to the student's explanation in Part B. We wanted to see if teachers could move beyond thinking of slope in terms of one-unit changes in x. Part B of "Slope and Division" gives teachers a chance to extend their meanings for slope to situations where x does not change by one, or alternatively reveal the limitations of their meanings for 3.04.

Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04. Convey to Mrs. Samber's students what 3.04 means. **Part B.** Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04. A student explained the meaning of 3.04 by saying, "It means that every time *x* changes by 1, *y* changes by 3.04." Mrs. Samber asked, "What would 3.04 mean if *x* changes by something other than 1?" What would be a good answer to Mrs. Samber's question?

Figure 1. The item "Slope and Division" was designed to reveal meanings for slope. © 2014 Arizona Board of Regents. Used with permission.

Rubric Development

After the first round of data collection from 144 teachers in Summer 2012 we categorized the thinking revealed in the items using a modification of a grounded theory approach (Corbin & Strauss, 2007). The modification is that we began our data analysis with strong theories of understanding magnitudes and rates of change, and of the nature of mathematical meanings and of characteristics that make them productive in instruction. We developed rubrics by grouping grounded codes into levels based on the quality of the mathematical meanings expressed.

We read the teacher's response literally, asking, "If this is what they said to a class, what meanings for the mathematical idea might students' learn?" During team discussions of rubrics and responses, we continually asked ourselves. "How productive would the teacher's response be for a student if this is what she or he said while teaching?" and, "How might students understand what the teacher said were they to take it at face value?"

The summary rubric for Slope and Division is given in Table 1. The rubric was refined many times as the project team conducted multiple rounds of scoring on data collected in Summer 2013.

Table 1. Rubric for Part A of "Slope and Division."

Level A3 Response:	The teacher conveyed that *x* can change by any amount and that *y* changes by 3.04 times the change in *x*.
Level A2a Response:	***Any*** of following: – The teacher wrote that for every change of 1 in *x*, there is a change of 3.04 in *y*. – The teacher wrote that for every change of 2.7 in *x*, there is a change of 8.2 in *y*. – The teacher wrote that a difference in *x* values is compared to a difference in *y* values.
Level A2b Response:	The teacher conveyed in words or graphically that the slope gives information about how to move horizontally and vertically. For example: – If *x* moves to the right 1 space, *y* moves up by 3.04. – If *x* runs by 2.7, *y* rises by 8.2. – The slope tells us to move horizontally by one and vertically by 3.04.
Level A1 Response:	***Any*** of following: − The teacher conveyed that 3.04 is the result of a calculation. − The teacher used a phrase such as "average rate of change", "constant rate of

	change" or "slantiness" without addressing the question of how 3.04 relates changes in x and changes in y. – The teacher simply stated the idiom "rise over run" without describing the changes.

Level A3 responses convey a multiplicative meaning for slope. A multiplicative meaning for slope builds on the meaning for quotient as a measure of relative size. Level A2a and level A2b responses convey an additive or chunky meaning for slope. Level A2a responses are considered slightly more productive for students than A2b responses because the meaning of slope in Level A2a responses is not constrained to horizontal and vertical motion on a Cartesian graph, but could be used productively in real world situations. Level A1 responses on our rubric represented more than one possible meaning for slope, but each of these meanings are similar in the sense that they convey that the meaning of slope is something to memorize. We scored responses that did not fit any other category at level A0. In cases where teachers responded with multiple meanings for slope in one response we decided to categorize their response according to the highest level meaning they conveyed.

Table 2. Categorization for Part B of "Slope and Division."

Gave reasonable meaning for 3.04	The teacher gave a mathematically reasonable explanation of what 3.04 means. For example "3.04 is the ratio" or "3.04 tells us how many times as large Δy is as Δx."
Gave explicit computations to find Δy	The teacher gave a clear instruction to find the change in y, the change in x should be multiplied by 3.04.
Gave vague computations to find Δy	The teacher answered the question "how to you find the change in y?" but does so without explicitly mentioning the change in y. For example they said "multiply it by 3.04."

The purpose of Part B is to allow teachers to think about the change in x varying continuously instead of in jumps of a fixed amount.

Briefly, common responses to Part B included explaining what 3.04 means, explaining how to find the change in y given an arbitrary change in x, or giving an example of how much y would change by if x changed by two. The quality of responses in each category varied from teachers who gave clear and understandable explanations of the meaning of 3.04 to those who explained what 3.04 meant by saying only "multiply it by 3.04." In scoring Part B we noted mathematical mistakes such as confounding y with Δy in a separate score not reported here. Although a portion of the responses at each level do contain mathematical errors, we categorize responses by the primary meaning conveyed ignoring mathematical mistakes. The categorization in Table 2 is based on our rubric for Part B. We will give the rubric for Part B and data in the longer paper.

Administration and Scoring

We administered the *MMTsm* to 157 high school teachers in two different Southwestern cities in Summer 2014. The high school teachers took the diagnostic exam at the beginning of professional development programs. The first author scored all responses to "Slope and Division." To estimate interrater reliability (IRR) an outside collaborator scored 50 overlapping responses. Scorers had perfect agreement on 84% of responses to Part A and 72% on responses to Part B. Non-perfect agreement was scored as disagreement. These IRR scores are lower than most other items on the *MMTsm* due to the complexity of teachers' responses. Responses were often not written in complete sentences and used pronouns with unclear antecedents so it was difficult to determine whether or not a student could make sense of the teacher's explanation. Because of the probability that the scorers might pick the same level by chance we also computed Cohen's Kappa for Part A (.773) and Part B (.621).

Results

The most common meaning conveyed in our sample was a chunky, additive meaning for slope (See Table 3).

Table 3. Responses to Part A "Slope and Division."

Response	Math Majors	Math Ed Majors	Other Majors	Total
A3-relative size	0	0	3	3
A2a-chunky	12	8	21	41
A2b-chunky graphical	19	29	30	78
A1-memorized	4	11	13	28
A0-other/IDK	1	1	2	4
No response	0	0	3	3
Total	36	49	71	157

Only three teachers out of 157 used a multiplicative meaning for quotient in explanations of slope in Part A. Approximately 76% of teachers showed a chunky or additive meaning for slope. Interestingly, about 86% of teachers who majored in mathematics and 82% of teachers who majored in mathematics education answered a chunky, additive meaning. Although chunky meanings for slope can be used productively in some situations, the responses to Part B often indicated that the teachers struggled to extend their chunky meaning to situations where the change in x is not equal to one. The response in Figure 2 conveys that slope gives information about how to move vertically and horizontally on a graph. The response conveys a chunky meaning for slope because the changes occur in chunks of one and 3.04. The meaning of 3.04 seems to be tightly tied to the change in y and only loosely connected to the associated change in x.

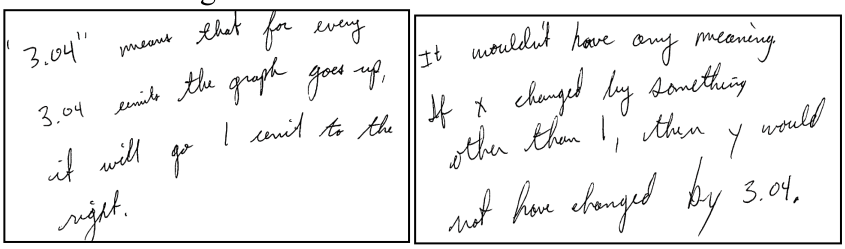


Figure 2. One teacher's "chunky" response to Part A and B.

The Part B response provides confirmation that for this teacher 3.04 is more strongly associated with the change in y, then a comparison of the relative size of the change in y and a change in x. It might convey to students that the slope gives information about vertical and horizontal motion on the graph and that the number 3.04 is only associated with the change in y and not with a comparison of changes in x and y.

The response in Figure 3 conveys that the slope is strongly associated with the change in y. In this case, the response incorrectly confounds the change in y with the slope. When the meaning for slope conveyed emphasizes that x changes by one the value of the slope and the change in y are identical and it becomes easier to confuse the two concepts.

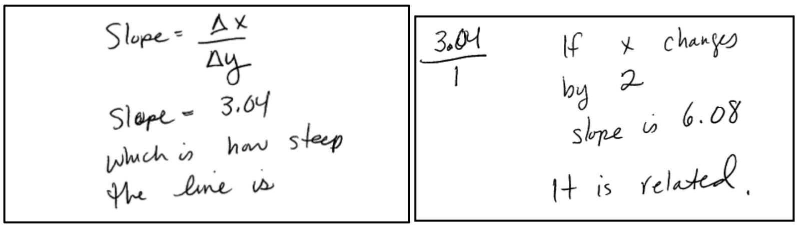


Figure 3. One teacher's response to Part A and associated chunky Part B response.

Some chunky responses conveyed that the only points on the line that "mattered" were the points obtained by the process of moving over and up in fixed chunks (see Figure 4). This response is not consistent with imagining that between any two points on the line there are infinitely many points.

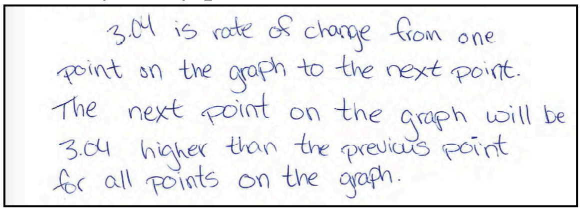


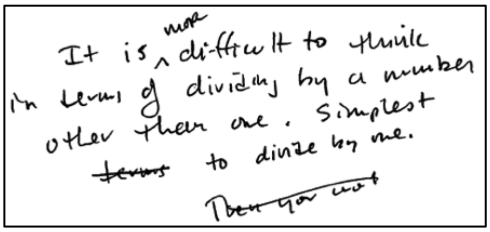
Figure 4. Chunky response conveying that the points on the line only occur at fixed intervals.

There are a variety of consequences of conveying that points on the line only occur at fixed intervals. If points only occur at fixed intervals it is possible to conceptualize slope as the distance between two points on a line. Some teachers in our sample explicitly responded that the slope is a distance between two points and some Calculus students who were interviewed on "Slope and Division" also told us explicitly that slope is the distance between the two points used in the slope formula. After confirming that, to the student, slope is a length, the interviewer asked the student, "Why do you divide the change in y and the change in x to get a length?" The student responded, "Because, it's you've got the one x here and the other one here and so you are trying to find the way which they both get to each other basically."

Table 4. Responses to Part A and Part B of "Slope and Division."

		Part B Response					
		Gave reasonable meaning for 3.04	Gave explicit computations to find Δy.	Gave vague computation s to find Δy.	Other	IDK Blank	Total
Part A	A3-relative size	1	2	0	0	0	3
	A2a/A2b chunky	30	31	18	40	0	119
	A1-memorized	2	8	6	8	4	28
	A0-other	2	0	0	0	1	3
	IDK/blank	0	0	1	0	3	4
	Total	35	41	25	48	8	157

In the case of the 48 responses categorized as "other" it was clear that the teacher struggled to respond to a situation with a change of x not equal to one. Note that 40 out of the 119 teachers who conveyed a chunky meaning in Part A were unable to cope with Part B in even a limited way.



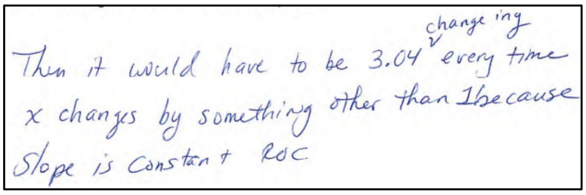


Figure 5. Two teachers (who conveyed chunky meanings in Part A)' responses to Part B

The two teachers in Figure 5 wrote chunky meanings in Part A and had difficulty explaining what 3.04 means when x changes by something other than 1. This is evidence that holding a chunky meaning for slope does not necessarily enable a teacher or learner to understand the proportional relationship between changes in x and changes in y.

Conclusions

The results show that many teachers have chunky meanings for slope that do not appear to be connected to an image of the relative size of Δx and Δy. If their meaning for slope was based on an understanding of the relative size of Δy and Δx, it should be easy to note that Δy is always 3.04 times as large as an arbitrary Δx in Part B. An inability to deal with an arbitrary sized Δx is problematic because in Calculus Δx becomes arbitrarily small yet retains a relationship of relative size with Δy. Although the results are not from a nationally representative sample of teachers, the sample size is large enough to strongly suggest that Stump's (1999;2001) and Coe's (2007) descriptions of a few teachers' meanings for slope are apparent in a much larger sample of teachers. Further investigations could use this instrument to research a nationally representative sample of teachers.

References

Ball, D. L. (1989). *The Subject Matter Preparation of Prospective Mathematics Teachers: Challenging the Myths*. Retrieved from East Lansing, MI:

Ball, D. L., & McDiarmid, G. W. (1989). The Subject Matter Preparation of Teachers. Issue Paper 89-4.

Byerley, C., & Hatfield, N. (2013). *Pre-Service Secondary Teachers' Meanings for Fractions and Division*, Denver, Colorado.

Castillo-Garsow, C. (2010). *Teaching the Verhulst Model: A Teaching Experiment in Covariational Reasoning and Exponential Growth.* ARIZONA STATE UNIVERSITY. Available from Google Scholar

Castillo-Garsow, C. W. (2012). Continous quantitative reasoning. In R. Mayes, R. Bonillia, L. L. Hatfield, & S. Belbase (Eds.), *Quantitative reasoning and Mathematical modeling: A driver for STEM Integrated Education and Teaching in Context. WISDOMe Monographs* (Vol. 2). Laramie, WY

: University of Wyoming Press.

Coe, E. E. (2007). *Modeling Teachers' Ways of Thinking About Rate of Change.* Arizona State University. Available from Google Scholar

Coper-Gencturk, Y. (2015). The Effects of Changes in Mathematical Knowledge on Teaching: A Longitudinal Study of Teachers' Knowledge and Instruction. *Journal for Research in Mathematics Education, 46*(3), 280-330.

Corbin, J., & Strauss, A. (2007). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory* (3rd ed.): Sage Publications, Inc.

Glasersfeld, E. v. (1995). *Radical constructivism: a way of knowing and learning*: Psychology Press.

Stump, S. (1999). Secondary mathematics teachers' knowledge of slope. *Mathematics Education Research Journal, 11*(2), 124-144.

Stump, S. (2001). Developing preservice teachers' pedagogical content knowledge of slope. *The Journal of Mathematical Behavior, 20*(2), 207-227. doi:10.1016/S0732-3123(01)00071-2

Thompson, P. W. (2013). In the absence of meaning… *Vital directions for mathematics education research* (pp. 57-93): Springer.

Thompson, P. W. (2015). Researching mathematical meanings for teaching. In L. D. English & D. Kirshner (Eds.), *Third Handbook of International Research in Mathematics Education.* London: Taylor and Francis.

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. J. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe, & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing. WISDOMEe Monographs* (Vol. 4, pp. 1-24). Laramie, WY: University of Wyoming.