Transforming graduate students' meanings for average rate of change

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This report offers a brief conceptual analysis of average rate of change (AROC) and shares evidence that even mathematically sophisticated mathematics graduate students struggle to speak fluently about AROC. We offer data from clinical interviews with graduate teaching assistants who participated in at least one semester of a professional development intervention designed to support mathematics graduate students in developing deep and connected meanings of key ideas of precalculus level mathematics as part of a broader intervention to support mathematics graduate students in teaching ideas of precalculus mathematics meaningfully to students. The results revealed that the post-intervention graduate students describe AROC more conceptually than their pre-intervention counterparts, but many still struggle to verbalize a meaning for AROC beyond average speed, a geometric interpretation based on the slope of a secant line, or a computation.

Key words: graduate student teaching assistant, average rate of change, precalculus

It may seem natural to assume that a graduate student in mathematics possesses strong meanings of foundational mathematics ideas because of their extensive experience studying mathematics. However, Speer (2008) and colleagues (Speer, Gutmann, & Murphy, 2005) reported that completion of more advanced mathematics courses does not necessarily improve a teacher's understandings and teaching practices. Other studies have advocated that teachers need productive meanings of the ideas they intend to teach (Carlson & Oehrtman, 2009; Moore, et al., 2011). Thompson, Carlson & Silverman (2007) claimed that:

> If a teacher's conceptual structures comprise disconnected facts and procedures, their instruction is likely to focus on disconnected facts and procedures. In contrast, if a teacher's conceptual structures comprise a web of mathematical ideas and compatible ways of thinking, it will at least be possible that she attempts to develop these same conceptual structures in her students. We believe that it is mathematical understandings of the latter type that serve as a necessary condition for teachers to teach for students' high-quality understanding (pp. 416-417).

In a recent study, Teuscher, Moore & Carlson (2015) report that a teacher's mathematical meanings provide a lens through which a teacher makes sense of student thinking. The teacher's model of students' thinking influences her subsequent instructional actions, including the nature of her questions, her questioning patterns, and the quality of the discussion she leads.

In a context of an intervention to support mathematics graduate students to act in productive pedagogical ways, we engaged graduate students preparing to teach precalculus at the college level in completing tasks aimed at developing their meanings of key ideas of precalculus level mathematics that are foundational for learning calculus. This study investigated mathematics graduate students' meanings of average rate of change (AROC) for the purpose of understanding graduate students' pre-intervention meanings of AROC and the degree to which our interventions impacted their meanings of this idea.

We describe what research has reported to be a productive meaning for the idea of AROC (Thompson, 1994). We follow this with a description of our intervention for shifting graduate mathematics students' meaning for AROC and conclude by reporting on results that point to the distinctions between pre- and post-intervention participants' meanings for AROC, and reveal the varied fluency among participants in *speaking with meaning* about AROC when probed in a clinical interview setting (Clark, Moore, & Carlson, 2008).

Theoretical Framework

We view an individual's *expressed meaning of an idea* as the spontaneous utterances that an individual conveys about an idea. From these utterances we can make inferences about how an individual has organized her experiences with the idea. The meaning held by an individual is then the organization of the individual's experiences with an idea, also referred to as a scheme. It is through repeated reasoning and reconstruction that an individual constructs schemes to organize experiences in an internally consistent way (Piaget & Garcia, 1991; Thompson, 2013; Thompson, Carlson, Byerley, & Hatfield, 2013). For example, an individual's meaning for the idea of *average rate of change* might consist of the calculation for the slope of a secant line, or simply $\Delta y/\Delta x$. An individual who has committed to memory that the average rate of change is the slope of a secant line does not possess the same meaning as someone who sees the slope of a secant line as the constant rate of change that yields the same change in the dependent quantity (as some original non-linear relationship) over the interval of the independent quantity that is of interest. These two individuals hold different meanings for the same idea, and the consequences of such differences can be profound.

An individual's meaning for the idea of *average rate of change* can be further developed through reflection, which occurs when she is faced with perturbations to her current meanings for average rate of change (Dewey, 1910).

A productive meaning for the idea of average rate of change

Constructing a rich meaning of *average rate of change* entails conceptualizing a hypothetical relationship between two varying quantities in a dynamic situation. Given a relationship between the independent quantity A and the dependent quantity B, and a fixed interval of measure of quantity A, the average rate of change of quantity B with respect to quantity A is the constant rate of change that yields the same change in quantity B as the original relationship over the given interval. In order to understand this complex idea meaningfully, an individual must first conceptualize the idea of *quantity* as a measurable attribute of an object (e.g., an airplane's height in feet above the ground, number of minutes elapsed since noon). Next, provided a situation in which two quantities vary in tandem, an individual must develop an understanding for what it means to describe the rate of change of one quantity relative to the other. Namely, the individual must conceptualize the multiplicative comparison of changes in the two quantities (the change in the output quantity is always some constant times as large as the change in the input quantity). In the special case that the relative size of changes in one quantity relative to the other remains constant, we say the quantities vary with a constant rate of change (CROC). Individuals with a robust meaning will draw connections between AROC and CROC and view those two connected ideas as a means for approximating values of varying quantities in dynamic scenarios.

The Intervention

The graduate students initially participated in a 2-3 day workshop in which they completed mathematical tasks that were designed and sequenced to support graduate students in constructing a productive meaning for the idea of *average rate of change*[1]. The graduate

[1] The idea of *average rate of change* is the culminating idea of the first instructional unit of the precalculus level course the graduate students will be teaching during the upcoming semester.

students confronted problems and questions designed to perturb their expressed meanings for AROC. The intent was to prompt reflection and subsequent shifts in their meanings of this idea. Concurrent with teaching the course during the fall or spring semesters, the graduate students attended weekly 90-minute seminars. The main goals of the weekly seminars were to support the graduate students in developing more productive meanings of the key ideas to be taught during the upcoming week, and to support them in clearly explaining their meanings for those ideas to others. As part of the work towards these goals, each of the graduate students decided on a lesson implementation plan detailing how they would engage their students in achieving their learning goals for that week prior to each seminar meeting. Participants further prepared mini-presentations of the material to practice talking about difficult or novel ideas. When preparing, participants used the Pathways Precalculus curriculum (Carlson, Oehrtman, & Moore, 2015), a research-based curriculum that incorporates student thinking and scaffolds the development of key ideas. The Pathways curriculum included materials designed to advance participants' understanding of AROC when preparing to use the student materials in their own classrooms.

Methods

We collected data from mathematics graduate students and instructors at three large, public, PhD-granting universities in the United States. Participants' teaching experience varied between zero and 11 years, at both the K-12 and tertiary level. We conducted semi-structured clinical interviews with 21 graduate teaching assistants, all of whom had at least one semester experience teaching the Pathways Precalculus course as lead instructor or recitation leader (Clement, 2000). Interviews addressed multiple issues, ranging from perceived shifts in beliefs about the roles of students and teachers to understandings of mathematical ideas to descriptions of teaching practices and goals. The lead author conducted these interviews, recording each using both a video camera and Livescribe technology to capture audio-matched written responses to sample teaching scenarios provided during the interviews. Interviews lasted 1-2 hours, and were transcribed and coded by three members of the research team. Members of our team analyzed videos in pairs at first, identifying themes of interest relative to our conception of a productive expressed meaning for AROC before working individually to continue coding and reconvening as a group to discuss our findings (Strauss & Corbin, 1990).

Results

We first share data that reveals the common meanings that the graduate students held for the idea of AROC when entering the program. We then share excerpts from clinical interviews with experienced participants to reveal the varied fluency participants demonstrated when describing the idea of AROC. The results show that the meanings for AROC conveyed by the novice and experienced groups are, in fact, categorically different; however, not all experienced participants shifted to speak fluently about the idea of AROC.

Pre-Intervention Meanings for AROC

As a warm-up activity for the start of a Summer 2015 teaching assistant workshop, we asked seven math graduate students to describe the meaning of "average rate of change." Each participant's response is recorded in Figure 1. Their responses align with the authors' prior experiences with both students and teachers at the secondary and tertiary levels; most of the participants provided geometric interpretations based on imagining a secant line between two points on the graph of a function. In particular, we see that Alan[2] described AROC both computationally (i.e., $\Delta y/\Delta x$) and geometrically as a line, instead of highlighting the key attribute of the line—its slope. Brian did mention slope, though he did so while conveying the idea that slope is an amount of change in the dependent quantity for each unit change in the independent quantity, a somewhat restrictive meaning for slope as it fails to support reasoning about variation when changes in the independent quantity have magnitude other than 1. Cassie and Diane spoke explicitly about steepness, a visual aspect of a graph that is simultaneously restricted to the Cartesian coordinate system and, in that setting, is potentially misleading when the coordinate axes do not have the same scale. Edgar provided two equivalent descriptions of how to *compute* the AROC over a given interval, but did not communicate what the result of that computation would *represent*. Greg commented on the "predictive" quality of AROC, making him the only participant to explicitly highlight the idea that AROC provides an alternate means for characterizing how two quantities change together. This thinking, however, is missing many elements of what we have previously characterized as a productive meaning for AROC.

Responses to the question: What does "average rate of change" mean to you?	
Alan:	A straight line between two points on a graph
Brian:	As one variable changes for every one unit, how much is the other variable changing. Slope.
Cassie:	Steepness of a graph, like how steep or how flat it is.
Diane:	Steepness of a graph. […] Uh, I don't have actual words.[…] Slope or derivative.
Edgar:	Rate of change over an interval
Frank:	The amount the dependent variable changes divided by the amount the independent variable changes. Delta y divided by delta x.
Greg:	I lost all the words…It's the predictive effect of changing one variable and the amount and how it's going to affect the other variable. One quantity affecting change in another quantity.

Figure 1. Pre-intervention participant descriptions of AROC

While most of the responses were accurate statements about AROC, the participants' expressed meanings were predominantly geometric or computational; moreover, only one of the seven participants spontaneously hinted at the idea that the AROC serves as a tool of approximation for rates of change within dynamic scenarios.

Post-Intervention Meanings for AROC

We have analyzed 11 clinical interviews with participants who experienced at least one summer workshop and one semester of our intervention. In contrast to the predominantly geometric and computational descriptions of AROC from our pre-intervention participants, 7 of the 11 participants attempted to describe a conceptual meaning for AROC. These descriptions can be classified as: the productive, general meaning described in our theoretical framework; a special case of that meaning for average speed; or, in one instance, a distinct interpretation the participant called "linearization." The other four participants offered explanations that fall into the last four categories described in Table 1.

[2] Psuedonyms are used throughout the reporting to protect the identity of participants.

Table 1. Experienced participant descriptions of AROC

Expressed Meaning Category	*Sample Excerpts from Clinical Interviews*	*Number of Instances**
Productive – General	[Students] have to understand constant rate of change because the average rate of change is the constant rate of change someone else would have to go, and I'm talking about average speed now, to achieve the same change in output for a given change in input. So, if you don't have meaning for constant rate of change, well, then average rate of change is just this number.	4
Average Speed	[AROC] is a constant rate of change for that specific time and distance, or uh, you know how I mean…	6
Conceptual Other	I would like to say linearization. Right, this idea of approximating something that isn't linear in a linear fashion.	1
Computational	… this final minus initial over the outputs and this final minus initial over the inputs and that's a rate.	2
Geometric	Average rate of change is the constant rate of change to go between two points.	2
Incorrect	I want my students to understand that constant rate of change is a special case, I guess of average rate of change. It's this special case that exists when the corresponding changes in our two quantities are proportional.	2
None		1

* Total exceeds 11 because some interviewees conveyed more than one expressed meaning.

The excerpts in Table 1 highlight the fact that the impact of the intervention on participants is far from uniform. One participant failed to provide a clear statement of a meaning for AROC as he talked around the issue for 14 minutes during his interview. We found this surprising in light of the fact that this participant had taught the idea of AROC from conceptually oriented materials for the past 4 semesters. Two of the six participants who mentioned average speed did not convey a meaning for AROC beyond the context of comparing distance and time. The sample excerpt for an incorrect meaning suggests that the participant developed a meaning for AROC linked to CROC in a non-standard way; conventional treatment of the two ideas typically describes AROC as a CROC approximation instead of viewing CROC as a special case of AROC. Yet another participant proclaimed, "I will forever think of average rate of change as the slope of the secant line." The fact that these participants did not immediately produce the meaning for AROC supported both by the intervention and the curriculum materials points to the complexity of the idea of AROC and the difficulty that even graduate students had in modifying their strongly held geometric and computational images of the idea of AROC to a more robust scheme with connections that are rooted in a conceptual meaning that can be expressed in multiple representational contexts.

Nonetheless, many participants' expressed meanings did align with our productive meaning for AROC, even if only as the special case of average speed, focusing on a relationship between varying quantities. During the intervention, leaders encouraged participants to “speak with meaning” as a tool to support their students in reasoning about quantities. To “speak with meaning” means to use appropriate language, describe the underlying meanings of specialized vocabulary (e.g., explains “proportional” instead of just using that word), and offer multiple ways of explaining a concept. Consider the “Productive Meaning” excerpt from Table 1 that was conveyed by a participant with 3 years of experience with the intervention, first as a participant and more recently as a leader. Not only did she express a productive meaning for AROC, using appropriate descriptions that highlighted *changes* in quantities as opposed to *values* of quantities, she further made explicit the connection between CROC and AROC and described the mental imagery she hopes her students develop. She later elaborated the importance of students imagining a second object or scenario that displays a CROC relationship that would yield the same change in output over the given interval of the input quantity.

Similarly, Hannah stumbled slightly, but ultimately described AROC in terms of changes in quantities, as seen in the following:

> *I think one needs to understand that average rate of change means that […] two quantities are varying but not necessarily at a constant rate of change—like the output quantity can, umm, not have a constant factor with respect to the input quantity. But the average rate of change of that relationship would be like if the…if there was a constant rate of change, the same output would be covered for a given amount of input. I think the easiest one for students to understand with that is like the example of like distance and speed. So if you're driving your car at a constant speed and I am like stopping and going and slowing down and speeding up, we will cover the same amount of distance in the same amount of time. And your—the constant rate that you go—is the same with my average rate. But I find that with that example it's really […] hard for students to talk about things not in terms of time. I also find that using the word "average" is confusing to students.*

She continued to reflect on a driving context as a familiar example to support students’ reasoning about AROC, but demonstrated an awareness of student thinking by highlighting that particular example as potentially problematic for students to generalize beyond contexts dependent on time. She also expressed an awareness of student difficulties with the multiple meanings of the word “average” appearing in the phrase *average rate of change*. Interestingly, though Hannah demonstrated a relatively high level of fluency in speaking with meaning about AROC, she pointed out that this particular idea is usually difficult for her to discuss with her students, saying:

> *I was struggling with it, and […] it's just hard to word it in terms of input and output and varying quantities without having a concrete example. And so, to me I'm not even sure that [students are] not getting it so much as that they're not able to articulate it.*

Discussion

The vast majority of participants held weak meanings for the idea of AROC at the beginning of the study. Our findings further revealed that our interventions were only moderately effective in supporting the graduate students to acquire productive meanings for the idea of AROC. The initial impoverished meanings expressed by graduate students were widespread across all three institutions, suggesting that that the issues involved in shifting graduate students’ meanings are not unique and require further investigation. These findings challenge the assumption that graduate students in mathematics have strong meanings of fundamental ideas of mathematics. Failure to act on this faulty assumption may have severe consequences for improving the predominantly procedural focus that exists in many introductory undergraduate courses in colleges and universities across the US.

Graduate mathematics students who hold a meaning for AROC that is strictly geometric (i.e., slope of secant line) will be unable to support their students in developing a quantitative meaning for AROC that could be leveraged to build ideas of accumulation from rate of change foundational to applications of calculus. Moreover, though several of the participants commented on how the Pathways materials exposed them to new ways of thinking about the mathematical ideas, these new ways of thinking do not necessarily translate to what the participants have as goals for their students' learning.

We also note that, though not described here, our experiences in working with the graduate students during the interventions produced encouraging anecdotal evidence that the opportunity to reconceptualize fundamental ideas may have a lasting impact on their image of what effective mathematics teaching entails. This leaves us optimistic that ours and other similar efforts might motivate mathematicians to engage in work to make undergraduate mathematics instruction more meaningful for students.

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