Symbolizing and Brokering in an Inquiry Oriented Linear Algebra Classroom

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The purpose of this paper is to explore the role of symbolizing and brokering in fostering classroom inquiry. We characterize inquiry both as student inquiry into the mathematics and instructor's inquiry into the students' mathematics. Disciplinary practices of mathematics are the ways that mathematicians go about their profession and include practices such as conjecturing, defining, symbolizing, and algorithmatizing. In this paper we present examples of students and their instructor engaging in the practice of symbolizing in four ways. We integrate this analysis with details regarding how the instructor serves as a broker between the classroom community and the broader mathematical community.

Key words: symbolizing, brokering, inquiry, linear algebra, disciplinary practices

Creating and sustaining engaged classrooms in which students learn particular mathematics and develop positive mathematical dispositions that transcend course-specific concepts is a daunting and challenging endeavor. For instructors, these challenges include (a) creating or selecting tasks that afford opportunities for students to learn mathematics by doing mathematics, (b) leading and facilitating small group and whole class discussions in which student ideas are shared and valued, and (c) relating students' intuitive, informal, or blossoming ideas to conventional and more formal mathematics. We refer to classrooms where these challenges are realized as "inquiry-oriented." Prior research (e.g., Laursen, Hassi, Kogan, & Weston, 2014; Rasmussen, Kwon, Allen, Marrongelle, & Burtch, 2006) points to the power of engaging students in typical mathematical practices through inquiry in undergraduate mathematics. As such, the need exists to further investigate and understand the relationships between what students do, what the instructor does, and the role of tasks in these inquiry-oriented classrooms. This report, using symbolizing in an inquiry-oriented linear algebra classroom as a case study, makes a contribution toward this need.

Literature and Theoretical Framing

We operationalize the notion of inquiry, using the definition put forth by Rasmussen and Kwon (2007), both in terms of what students do and what instructors do in relation to student activity. On the one hand, students learn mathematics through inquiry as they work on challenging problems that engage them in typical mathematical practices, which we refer to as disciplinary practices. Disciplinary practices of mathematics are the ways that mathematicians go about their profession and include practices such as defining, theoremizing, symbolizing, and algorithmatizing (Rasmussen, Wawro, & Zandieh, 2015; Rasmussen, Zandieh, King, & Teppo; 2005). On the other hand, instructors engage in inquiry by listening to student ideas, responding to student thinking, and using student thinking to advance the mathematical agenda of the classroom community (Rasmussen & Kwon, 2007).

Brokering

In addition to characterizing what constitutes disciplinary practices, over the years we have also developed and refined the work of instructors in leading inquiry-oriented classrooms (e.g., Rasmussen & Marrongelle, 2006; Rasmussen, Zandieh, & Wawro, 2009; Wawro, 2014; Zandieh

& Rasmussen, 2010). As part of this work, we adapted the idea of broker from the communities of practice literature (Lave & Wenger, 1991; Star & Griesemer, 1989; Wenger, 1998) to help make sense of the difficult work of inquiry-oriented instruction. By definition, a broker is someone who can facilitate communication and fluidity of practices between different communities and who has membership status in the different communities. Here we consider two different communities: the local classroom community and the broader mathematical community. Typically, the instructor has membership status in both communities. More importantly for us, brokers link practices (in our case defining, conjecturing, proving, etc.) between communities and are able to promote learning by introducing into the classroom community elements of practice from the broader mathematical community.

In previous work, we examined the case of students reinventing a bifurcation diagram in a first course in differential equations and the role of the instructor in this process. This analysis revealed three different types of instructor brokering moves: creating a boundary encounter, bringing participants to the periphery, and interpreting between communities (Rasmussen et al., 2009). In this paper we highlight the first and third of these brokering moves.

Creating a boundary encounter refers when a broker (i.e., the instructor) sets up an indirect interface between the classroom community and the broader mathematical community. A boundary encounter involves a boundary object, typically a well-chosen task or sequence of tasks, that provides an opportunity for students to engage in one or more disciplinary practices. In the sections that follow we delineate features of such tasks in our case study that opened a space for students to engage in the disciplinary practice of symbolizing.

Interpreting between communities is a brokering move in which the instructor coordinates students' mathematics with the more conventional or formal mathematics of the broader mathematical community. This type of brokering move typically occurs when the instructor inserts notations, symbols, graphs, diagrams, or provides other information that enables students to transcend the idiosyncrasies of their local classroom community. Interpreting between communities is significant because it shows how the instructor can connect student thinking to the well-developed mathematical culture. Moreover, it facilitates the sense of ownership of ideas and belief that mathematics is something that can be reinvented and figured out.

Symbolizing

Not all classroom activity is characterized by participation in disciplinary practices, even in inquiry-oriented classrooms. For instance, classroom mathematical practices capture the emerging content-specific mathematical progress of a local classroom community (Cobb, 2000; Rasmussen et al., 2015), whereas disciplinary practices capture how that progress might reflect what professional mathematicians do that transcends specific content. Symbolizing is the disciplinary practice of creating and using symbols to communicate mathematical ideas. Symbols include graphs, diagrams, and analytic expressions such as letters, numbers, and vectors. By engaging in their own symbolizing, students act like mathematicians – notating processes and connections between ideas with shorthand expressions that allow for efficiency of processing.

In this paper we highlight symbolizing of the following four types that we have found to be prevalent in inquiry-oriented classes:

(S1) Notating steps in a calculation or process,
(S2) Stating a relationship between two or more mathematical objects,
(S3) Creating a connection across two different representations (notating in new symbolism what has already been explained or described in a different way), and

(S4) Creating a unifying inscription, a graphic or diagram that illustrates multiple relationships at once.

In the results section, we call explicit attention to these four types of symbolizing in the context of students' solving problems and communicating their reasoning.

Research Setting and Methods

Our research over the last decade in the teaching and learning of linear algebra has been grounded in the design-based research paradigm of classroom-based teaching experiments (Cobb, 2000). This involves a cyclical process of (a) investigating student reasoning about specific mathematical concepts and (b) designing and refining tasks that honor and leverage students' mathematical ideas towards accomplishing the desired learning goals (Gravemeijer, 1994; Wawro, Rasmussen, Zandieh, & Larson, 2013). One product of this design-based research is the Inquiry-Oriented Linear Algebra (IOLA) curricular materials, designed to be used for a first course in linear algebra at the university level. At present, three units comprise the IOLA materials: Unit 1: Linear Independence and Span (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012; Wawro et al., 2013); Unit 2: Matrices as Linear Transformations; and Unit 3: Change of Basis, Diagonalization, and Eigentheory. Many of the tasks in the IOLA materials are created to facilitate students engaging in experientially real task settings in such a way that their mathematical activity can serve as a basis from which more formal mathematics is developed.

The data presented in this paper come from a classroom teaching experiment in a first course in linear algebra during Fall 2014 at a large public mid-Atlantic university. The data sources were classroom videos that capture small-group work and whole-class discussion, as well as students' written work from class. In addition, the four walls of the physical classroom were almost entirely whiteboards, which the instructor took advantage of by encouraging students to work in their small groups together at the whiteboard; as such, another primary data source was photos of student and teacher work on classroom whiteboards.

In this paper we focus on the class's mathematical development through the first task of Unit 3, "The Stretching Task" (see Figure 1). The first task builds from students' experience with linear transformations in $\mathbb{R}^2$ to introduce them to the idea of stretch factors and stretch directions and how these create a non-standard coordinate system for $\mathbb{R}^2$. This is the beginning of a larger sequence in which students reinvent the diagonalization equation $A = PDP^{-1}$. We analyzed the data to determine what types of symbolizing the students and the instructor were engaged in. These became our four types of symbolizing (S1) – (S4). In addition we examined the data for instances of each of the three types of brokering moves (of which we found examples of two of the types). Analysis of this data allows us to illustrate examples of students engaging in the practice of symbolizing, and we integrate this analysis with details regarding how the instructor serves as a broker between the classroom community and the broader mathematical community.

Results

In this section we provide examples of both student and teacher symbolizing activity. This symbolizing activity falls into the aforementioned four categories: (S1) - (S4). We begin with a description of the task itself as a boundary object and then follow with a description of student and teacher use of the four types of symbolizing as they occur. Symbolizing serves as a means for students to record and communicate their inquiry into the mathematics. Communicating through symbolizing also serves as a means for brokering within and across different groups of students in the classroom as well as for brokering between the classroom community and the mathematical community.

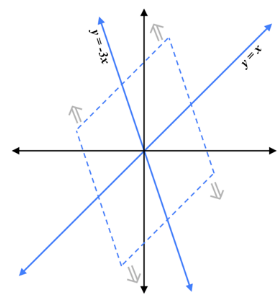
In Unit 3 Task 1 (referred to as "The Stretching Task"), students are asked to describe the result of a transformation given in terms of what the transformation does to two lines (See Figure 1). One goal of the task is to create a means for students to engage with ideas that will facilitate their learning about stretch factors and stretch directions and the possibility of using these stretch directions as a grid for their work in the plane. Although the formal definitions of eigenvector and eigenvalue arise later (Task 3 of this unit) for students, we purposely use the terms stretch direction and stretch factor here to immerse students in the geometric interpretation of these terms. In choosing this task, the instructor serves as a broker by presenting the students with a task that can serve as a boundary object between the classroom community and the mathematical community. The task serves as a boundary object in that it provides an opportunity for students to engage in the disciplinary practice of symbolizing in ways that begin to align with how the mathematical community uses symbolizing in this context.

The Stretching Task

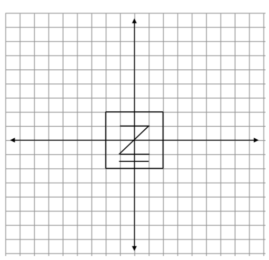
Imagine a linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that has the following properties:

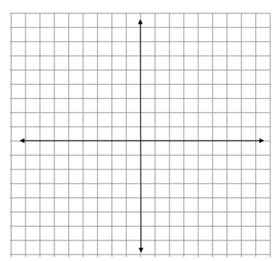
In the direction along the line $y = -3x$, the transformation stretches all points by a factor of two.

In the direction along the line $y = x$, the transformation keeps all points fixed.



1. Use the space on the right to sketch what should happen to the image shown on the left when it is stretched according to the transformation described above. You may use a combination of intuition or calculations, as well as any additional sketches below or on your group's whiteboard.





2. Determine what will happen to $\begin{bmatrix}2\\0\end{bmatrix}$ and to $\begin{bmatrix}-2\\2\end{bmatrix}$ under this transformation. Use an initial estimate from your sketch in problem 1. Then try to do a calculation that will determine these locations more precisely.

3. Determine a matrix that allows you to calculate what happens under the transformation to any point on the plane. Use it to check your sketch or improve its accuracy.

Figure 1. Task 1: The Stretching Task.

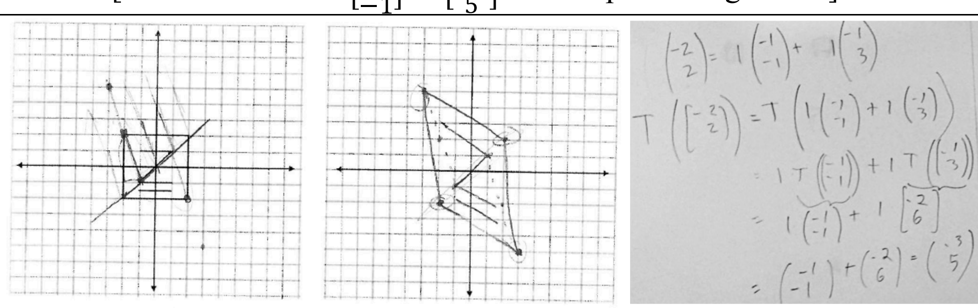
In what follows we present data from a particular classroom implementation of the materials in Fall 2014. Symbolizing of various types occurred during the parts of two class periods that students worked on Task 1. Student activity on Task 1 began with engagement within the graphical symbolization that they had been given in the introduction to the Stretching Task and in problem 1. This symbolizing is of types S1 and S2 in that students were notating steps in their graphing process and recognizing relationships that would help them create a transformation of the Z-box. Because this initial work occurred at the end of class, it was on the following class day that students and the instructor made connections between this graphical work on the task and other symbolic notation to describe the transformation (symbolizing type S3). In addition, the instructor introduced a unifying graphic based on student work to aid students in working with the transformation (symbolizing type S4). These symbolizing examples and the role of the teacher as a broker in these examples are detailed in the next four sections.

Within the graphical representation (S1 and S2)

Students initially engaged in the task by symbolizing within the graphical and verbal description that they had been given at the beginning of Task 1 and in Problem 1. Many students began by notating the points that stay fixed and then estimating the images of the points that stretch. Figure 2a illustrates the work of Donald, who presented at the board to explain his graphical symbolizing process (S1):

> *So you know the points along y = x are the same and, like, that'd be these points along that line. So you know like you get two of the corners, you know these points and these points are gonna stay the same. And then you also know that this stretches along the y = -3x line, which is like any of these. But this can be moved, like, kind of like a linear combination of this, where like you start along this line. And it stretches like up that way. And this corner point happens to, like, coincide with this point here which you know stays the same. So that's along the line y = -3x and then you just double it to get that point, which comes over here. And you do the same for down here [the lower right corner]. And then once you get the 4 corners, you can just like figure that everything else is gonna stretch kind of similarly.* [See Figure 2a]

From the video of that day of class we can reconstruct his explanation. First, he explained that the points along the line $y = x$, in particular the corner points $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, will "stay the same." He continued by noting the lines parallel to $y = -3x$ all stretch in the same way, away from their "start along this line," i.e., starting from the $y = x$ line. [See the line segments Donald drew in the left of Figure 2a]. Note also Donald's use of the phrase "linear combination." He did not explain the algebra of this phrase but symbolized this graphically as a stretch from the $y = x$ line along a line parallel to $y = -3x$. Donald continued by discussing the specific case of the corner point, $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$. He described that the corner point "happens to coincide with this point," i.e., the corner point $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ is on the same line segment as $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$. The line segment is one of those he drew parallel to $y = -3x$. Then "you just double it," i.e., double the segment from $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ to $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$. In doing this he marks the point $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$, which is the result of the transformation on the upper left corner of the box [See the line from $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ to $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$ in each part of Figure 2a.]



(a) (b)

Figure 2. (a) Donald's work on Problem 1 and *(b)* The instructor's record of Donald's work.

Donald's description exemplifies symbolizing in a graphical context to share steps in a process (S1) and show connections between pieces of graphical information (S2), i.e., how to combine the information that $y = x$ stretches by one and $y = -3x$ stretches by two. Clearly, verbalizations accompanied and were need to communicate his graphical approach. However, the focus remained on working within a graphically represented system of ideas. Next we discuss how other symbolizations allowed students to incorporate other ways of exploring this problem.

A transition to vector notation (S3)

On the next day of class, the instructor included a scanned copy of several examples of student work on this problem, including Donald's work in Figure 2a (and the work in Figure 3a and 3b in the next section). As a follow up to Donald's explanation, the instructor used a linear combination of vectors to record the student idea that $\begin{bmatrix}-2\\2\end{bmatrix}$ could be reached by going to $\begin{bmatrix}-1\\-1\end{bmatrix}$ and then travelling $\begin{bmatrix}-1\\3\end{bmatrix}$ (see the first line of Figure 2b). The students' idea that the vector $\begin{bmatrix}-1\\-1\end{bmatrix}$ stays fixed but $\begin{bmatrix}-1\\3\end{bmatrix}$ doubles under the transformation, T, is indicated by lines 2 though 4 of Figure 2b. Finally, the last line of Figure 2b indicates the fact that combining the fixed $\begin{bmatrix}-1\\-1\end{bmatrix}$ with the doubled $\begin{bmatrix}-1\\3\end{bmatrix}$ (now $\begin{bmatrix}-2\\6\end{bmatrix}$) reaches the point $\begin{bmatrix}-3\\5\end{bmatrix}$. The instructor's choice of symbols helps interpret between the student ideas and the standard mathematical notation. In particular, the symbols for a linear combination of vectors and the distributive properties of a linear transformation were familiar to the students from earlier work in the course, but they had not previously seen the application of a linear combination in the sense of line 2 of Figure 2b.

The symbolizing by the instructor connected the graphical reasoning of the student to a symbolic vector notation, creating a connection across representations (symbolizing type S3). In addition, the symbols written by the teacher served as what Rasmussen and Marrongelle (2006) define as a transformational record. Transformational records are "notations, diagrams, or other graphical representations that are initially used to record student thinking and that are later used by students to solve new problems" (Rasmussen & Marrongelle, 2006, p.389). In other words, they record student inquiry in a way that provides a symbolization for future inquiry.

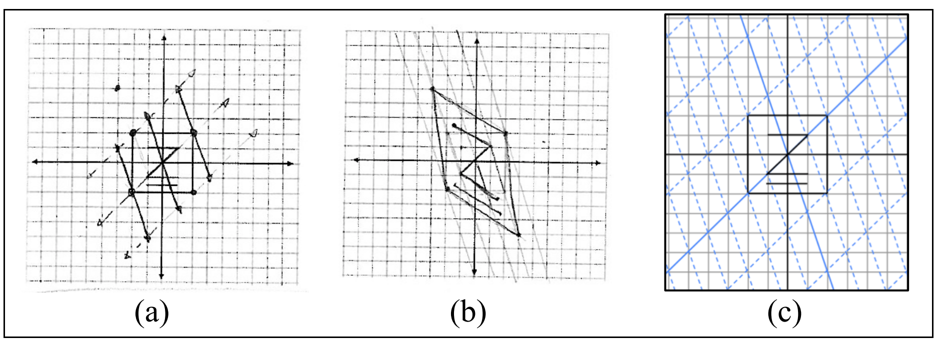


Figure 3. (a), (b) Two student examples showing a gridding of the plane using stretch directions. and *(c)* The instructor's graph of gridding using the stretch directions.

Creating a unifying graph (S4)

The instructor also included in her presentation examples from two other students on Problem 1 (Figure 3a, 3b). These examples show students creating a grid when trying to determine how the Z-box transformed. Note that each of these examples has similar features to

Donald's work, but more extensive gridding of the plane using lines parallel to $y = -3x$ and $y = x$. Each also has points marked at $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and at $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$ indicating the doubling of a vector along that line to find the new corner point.

The instructor emphasized the gridding in the student work and introduced the gridding of Figure 3c. In this way the instructor acted as a broker between the developing graphical symbolizing of the class and more sophisticated ideas from mathematics community.

The graphical representation in Figure 3c illustrates two ways to grid the plane. One (in grey) is the standard familiar grid and the other (in blue) is based on lines parallel to the stretch lines described in Task 1. This gridding can be used as a way to more easily see the doubling along the lines parallel to $y = -3x$. This is how Donald explained that $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ stretches to $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$, but now, with the complete grid available, that graphical method is available for any point. In addition to this practical result that connects to student thinking, the grid sets the stage for the student exploration in Task 2 of more sophisticated ideas of change of basis and diagonalization as described below. The graphic of Figure 3c then is unifying of students' current work and thus serves as an example of type S4 symbolizing. In addition, this graphic is key to the activity in Task 2 that leads to the creation of $A = PDP^{-1}$.

Conclusion

In this paper we explored student and instructor inquiry in the context of the disciplinary practice of symbolizing. Student inquiry in Task 1 involved exploring a graphical situation, creating symbols that expressed their emerging ideas about the mathematical situation, and symbolizing their graphical activity using vectors and vector equations. The inquiry involved: (a) creating or choosing appropriate ways to symbolize mathematical processes (S1) and relationships (S2) within particular representations, (b) making connections between different symbolizations of mathematical content (S3), and (c) creating a unifying graphical representation (S4). Each of these reflects a facet of the disciplinary practice of symbolizing, characterized through types S1-S4. Student inquiry into the mathematics of this task created a necessity for mathematical symbolism that students used to express their emerging ideas about the mathematical situation as well as to create more powerful and efficient solutions. The vector representations, vector equations, and unifying graphic (Figure 3c) provided additional tools for inquiry that were further developed and used in the subsequent tasks of this unit.

Instructor inquiry into students' mathematical thinking involves them making sense of and leveraging student insights so that they can appropriately connect the mathematics being developed by students in the classroom with the mathematics of the mathematical community. This is the notion of brokering. For the instructor to truly serve as a broker between the two communities, the instructor must participate in, recognize, and understand how students are engaging in the mathematics. A broker is not someone who simply relays information from one community to another, like a messenger; rather, a broker negotiates mathematical meaning between two communities. Furthermore, what tasks are selected and how they are used also plays a part in the brokering process, as tasks provide an interface through which the classroom community will encounter situations that can serve as a basis from which the more formal notions of the broader mathematical community can be developed. With the help of the instructor as broker, and boundary objects carefully chosen by the instructor, students can begin to act as mathematicians do. They can progress in their ability to engage in disciplinary practices in ways helpful not only for learning particular mathematical ideas but also for applying in other settings.

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