The mathematical symbol “dx” is a symbol for which there can exist different views about its characteristics, purposes, and roles. We wished to see how experts viewed the dx in a variety of settings. We chose four mathematical contexts and interviewed four mathematics professors in order to understand their various concept images of the dx. While there was little agreement among the experts’ responses, most of them did have a strong concept image that remained consistent throughout their interviews, despite our attempts to create cognitive conflict between the different mathematical contexts. We conclude that the existence of a range in the experts’ opinions is noteworthy, and that further study should be conducted in order to more fully explore this range and any implications for instruction that may result from it.

Key words: Calculus, Integrals, Differential Equations, Concept Image

In this preliminary report, we examine instructors’ views of some of the various roles of the symbol dx. This symbol can be found in a variety of mathematical settings, including definite and indefinite integrals, the formal definition for differential of a function, Leibniz notation for derivatives, the process of integration by substitution, and several types of ordinary differential equations. We wanted to see whether instructors’ concept images (Tall & Vinner, 1981) of this symbol would be consistent throughout all of these settings, or if different concept images would manifest given different settings.

Research has been done on how students perceive the dx in a definite integral. Sealey and Thompson (in press) summarized four such perspectives noted in the literature: a marker that points out the variable of integration (Artigue, Menigaux, & Viennot, 1990; Hu & Rebello, 2013; Jones, 2013; Nguyen & Rebello, 2011), a graphical width of a rectangle (Bajracharya, Wemyss, & Thompson, 2012; Wemyss, Bajracharya, Thompson, & Wagner, 2011), a small amount of a given physical quantity (Artigue et al., 1990; Hu & Rebello, 2013; Roundy, Manogue, Wagner, Weber, & Dray, 2015), or a difference or change in a quantity (Von Korff & Rebello, 2012). López-Gay, Martinez, & Martínez (2015) gave other concept images found in physics, including dx as an infinitesimal increment or linear estimate. While Hu and Rebello (2013) emphasized the importance of conceptualizing the dx as a width, Sealey and Thompson (in press) noted the importance of conceptualizing it as a difference in other mathematical contexts.

The purpose of our research is to explore further concept images of the dx in some additional mathematical contexts. Specifically, we chose to investigate experts' perspectives of the dx in definite and indefinite integration, the formal definition of the differential of a function, integration by substitution, and separable and exact ordinary differential equations. To create beginning points of reference, two surveys of textbooks were conducted. The first survey included nine books (Barnett & Ziegle, 1989; Breusch, 1969; Ellis & Gulick, 1988; Fisher & Ziebur, 1965; Hughes-Hallet, et. al., 2006; Mizrahi & Sullivan, 1982; Rees & Sparks, 1969; Stein, 1967; Stewart, 1987) which contained material found in traditional first- and second-semester calculus courses. We analyzed and compared any sections of these books in which definite integrals, indefinite integrals, differentials of functions, and integration by substitution were introduced and/or defined. The second survey included three books (Boyce & DiPrima, 2012; Stewart, 1987; Zill, 1997) that
contained information on basic ordinary differential equations, with which we compared and contrasted the sections that introduced separable and exact differential equations.

We found that only Hughes-Hallet et al. (2006) presented the idea that the $dx$ in the definite integral comes from the factor $Δx$ found in a Riemann Sum, while the other books stated that the $dx$ in a definite integral was merely notation with its only purpose to serve as a dummy variable that indicated the variable of integration. All of the books clearly stated the definition that, if $y$ were a function of $x$, the differential of $y$ is given by the formula $dy = y'(x)dx$, with all but one book stipulating that $x$ and $dx$ were independent variables, $x$ is any number in the domain of $y$, and $dx$ is any real number. Every book approached the evaluation of an integral that required substitution by the usual method of determining a $u(x)$ and using the relation $du = u'(x)dx$. However, there was no discussion as to the nature or roles of the various $dx$s that were seen throughout this process. Similarly, no matter the solution methods offered for separable or exact ODEs, no book explained the roles of the $dx$, nor discussed the permissibility of multiplying or dividing by $dx$ throughout the solution process.

**Theoretical Perspective and Methods**

Tall and Vinner’s (1981) concept image and definition were used to structure the design of the study and the analysis of the data. Because of the discrepancies and varieties of concept images and definitions found in the existing research and collections of textbooks, we wanted to interview experienced mathematics faculty to see if their concept images would be different, not only from the books’ images, but also from each other’s. We wanted to determine if their individual concept images would be more well-formed and align more closely to a formal concept definition than what the textbooks seemed to provide. Assuming the existence of well-formed concept images, we also wanted to see whether we might find some instances of potential cognitive conflict.

During a series of clinical interviews, four professors were shown a series of mathematical symbols, definitions, and situations in which the symbol $dx$ was present. Faculty members were chosen so that there was some variety in their research areas. Participants Sonya, Johnny, and Jackson each had research and/or teaching experience in analysis and differential equations, while Kurtis’ research areas included combinatorics and graph theory.

Prior to the interviews, we created an interview protocol, listing the order in which the aforementioned mathematical symbols, definitions, and situations would be presented to the subjects. Thirteen such symbols, definitions, and situations were divided into four categories, listed in Table 1. In addition to the question of how the subjects perceived the role of the $dx$ in each category, follow-up questions were posed if the subject stated something that differed markedly from the surveyed textbooks or other subjects’ responses. Johnny requested that his interview not be videotaped; thus his impressions have been taken from the authors’ notes. All of the other interviews were videotaped and later transcribed.

**Data and Results**

Since we were interviewing experienced instructors, it was possible that their individual concept images might have converged to a formal concept definition. But, as the textbooks did not show one formal definition but a variety of ways in which to think about the $dx$, we anticipated that the professors might not all have the same concept image. Data analysis is ongoing, but
Table 1

A Summary of Our Categories and Uses of $dx$

<table>
<thead>
<tr>
<th>Categories</th>
<th>Symbols, Definitions, or Situations Containing $dx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integrals</td>
<td>$\int f(x) , dx$, $\int_a^b f(x) , dx$, and $\int_a^b f(x) , dx$</td>
</tr>
<tr>
<td>Definitions and Notation</td>
<td>If $y = y(x)$, the notation $\frac{dy}{dx}$ and definition $dy = y'(x)dx$</td>
</tr>
<tr>
<td></td>
<td>If $x = x(t)$, the notation $\frac{dx}{dt}$ and definition $dx = x'(t)dt$</td>
</tr>
<tr>
<td>Integration by Substitution</td>
<td>$\int_1^4 \cos \sqrt{x} , dx$ versus $\int_2^4 \cos x , dx$, after $\int_1^4 \cos \sqrt{t} , dt$ used the substitution $dx = \frac{1}{2\sqrt{t}} , dt$</td>
</tr>
<tr>
<td>Two ODEs</td>
<td>1) The separable equation $\frac{dy}{dx} = g(y)h(x)$, and the solution steps $\frac{1}{g(y)} , dy = h(x) , dx$ and $\int \frac{1}{g(y)} , dy = \int h(x) , dx$</td>
</tr>
<tr>
<td></td>
<td>2) The exact equation $(2xy - 9x^2) , dx + (x^2 + 2y + 1)dy = 0$</td>
</tr>
</tbody>
</table>

Preliminary analysis indeed shows that while all four interview subjects were very consistent within their personal concept image, these images differed from one another, and did not align with a single formal concept image. Summaries of the subjects’ responses for each of our four contexts and some of the subjects’ personal concept images are given below.

**The $dx$ in Definite and Indefinite Integration**

Sonya and Jackson stated that the $dx$ in a definite integral comes from a limiting process applied to the width represented by the bases of Riemann Sum rectangles. Kurtis seemed to have a similar idea but was not as specific, saying that “it [the $dx$] comes from the $\Delta x$ [in a Riemann Sum]” without mentioning the image of rectangle widths. Johnny initially described the $dx$ as arising from the limit of “cuts in the interval between $a$ and $b$ on the $x$-axis,” but later changed his answer to “dummy variable” after some thought. All subjects except Kurtis also claimed that they viewed the $dx$ in an indefinite integral no differently than they viewed the $dx$ in a definite integral. Kurtis, however, claimed that the indefinite integral’s $dx$ had no meaning beyond being half of a notation (the other half being the integral sign) which signaled antidifferentiation.

**The $dx$ in Definitions and Leibniz Notation**

Kurtis said that $dy = y'(x) \, dx$ if and only if $\frac{dy}{dx} = y'(x)$, but did not feel that this meant that one could simply multiply or divide by $dx$ to go from one form to the other. Sonya agreed that such multiplication or division was not possible, while Johnny and Jackson had no problem with multiplying or dividing by $dx$ in this way. Another area of contention was that part of Johnny’s initial response when presented with the four notations and definitions in this section was to state...
an idea also found in Rees and Sparks (1969): the $dx$ in $dy = y'(x)dx$ could be either an independent variable or a function of two other variables, and that this definition of the differential of a function would still hold. Sonya and Jackson initially thought instead that $dx$ was strictly a dependent or independent variable depending on its position in the definition, but Sonya came around to Johnny’s view after that view was presented to her. Kurtis only went so far as to claim that the relationship between the $dy$ and $dx$ in $dy = y'(x)dx$ was the same as the relationship between the $dx$ and $dt$ in $dx = x'(t)dt$.

**The $dx$ in Integration by Substitution**

All subjects except Sonya seemed to feel that even though the substitution process began with the “$dx = x'(t)dt$” definition of the differential of a function, once the substitution was made, the $dx$ had transformed into a simple dummy variable, and was therefore now bereft of any deeper meaning. Jackson additionally mentioned that while the initial dummy variable (in $\int_{1}^{4} \cos \sqrt{x} \, dx$) and the transformed dummy variable (in $\int_{1}^{2} \cos x \, dx$) had similar roles as infinitesimal widths, we could still think of them as different, since one was the limit as $n$ goes to infinity of $\frac{4-1}{n}$ while the other was the limit as $n$ goes to infinity of $\frac{2-1}{n}$. This idea that the two versions of $dx$ have different sizes was also expressed by Sonya, but her image included an idea that both $dx$s were on different levels; specifically “macroscopic/microscopic” levels.

**The $dx$ in Separable and Exact ODEs**

Sonya felt that even though it may appear that we could multiply by $dx$ in order to separate variables in the separable equation, what is really happening instead is that we are multiplying by $\Delta x$ and then passing through the limit. Kurtis agreed with the idea that we are not really multiplying by $dx$, but seemed to think that it was always fine to proceed as if that is what were really happening. Johnny and Jackson, as before, had no problem with multiplying or dividing by $dx$. Similar responses occurred during the explanations of the exact ODE. Sonya was still uncomfortable with the idea of “moving the $dx$ around” but admitted that it is how solving differential equations is usually taught. Johnny and Jackson did not have this discomfort, and Kurtis declined to answer, stating that he was not as familiar with exact differential equations.

**Personal Concept Images**

While analyzing all of the subjects’ responses, it was found that Johnny, Kurtis, and Sonya seemed to have central images that ran throughout all of their answers. Johnny’s overall view seemed to be summarized by his initial response when presented with the two notations and definitions: “in some ways, they are all the same.” He noted that even though the traditional definition $dy = y'(x)dx$ came with the idea that $x$ and $dx$ were independent variables, there was nothing stopping us from assuming that $dx$ could also be a function depending on other independent variables (for example, $dy = y'(x)dx$ while $dx = x'(s)ds$), a view shared by Rees and Sparks (1969). If one were to continue this chain down to the last link, then the last differential in this chain can also fit that definition, as in our example: $ds = 1 \cdot ds$, since the derivative of $s$ with respect to $s$ is 1. He repeatedly said that these relations between differentials were “meaningful only in their relation to one another.” Thus, for example, one can multiply or divide by a $dx$ while manipulating an ODE, since that ODE also contains a $dy$, but in the “integration by
substitution” process, once the substitution has been made, the $dx$ becomes a dummy variable, since we no longer have a second differential.

Johnny’s central view of all $dx$s outside of integration having a numerical basis seems to run counter to Kurtis’ central view, which was that every instance of $dx$ or $dy$ was merely a product of a “useful notation” and had no mathematical meaning as a numerical entity. Kurtis said many times that all of these manipulations were products of a “perfectly good notation,” and thus easy for educators use when introducing concepts like the Chain Rule or integration by substitution, but while it may appear that mathematical operations with a $dx$ might be implied, Kurtis was adamant that this was not the case.

An image of Sonya’s was that she was uncomfortable multiplying or dividing by $dx$, since it was an infinitesimal quantity created by “passing $\Delta x$ through the limit.” To her, no matter the situation in which a $dx$ was being used in algebraic manipulations, the “real story” was that we were instead manipulating $\Delta x$ (a measureable quantity) and then passing through the limit, turning all $\Delta x$s into $dx$s. She noted that the convenience of simply saying “multiplying by $dx$” was helpful for instruction, but that we should be more careful about telling our students “we can multiply by $dx$.” Yet when it was presented to her, she seemed to accept Johnny’s and Rees and Sparks (1969) view that $dx$ could be an independent variable no matter the presentation. This might seem to contradict her idea that $dx$ was only some infinitesimal quantity unable to be manipulated. Further research will explore whether these two views are an example of cognitive conflict, or whether deeper questioning will lead to a more complete view of her total concept image.

Discussion

Many of the subjects’ responses suggest possible areas for future research. It is possible to use differentials to develop implicit differentiation (Mizrahi & Sullivan, 1982; Rees & Sparks, 1969) or generate derivatives by using differentials instead of taking the limit of a difference quotient (Dray, 2013; Dray & Manogue, 2010). Additional data collection could tell us if any of the subjects’ concept images allow or conflict with these developments. Sonya mentioned that the convenience of multiplying by $dx$ would not be appropriate with higher-order differentials; additional data collection could tell us if any of the subjects’ concept images agree or disagree. Several books gave differential rules that parallel derivative rules (an example being $d(uv) = u\, dv + v\, du$), and there are also proofs of the Chain Rule and various methods of finding the solutions to separable differential equations in which it appears that differentials are being multiplied, divided, or canceled. The belief in whether one could perform such manipulations with $dx$ divided our subjects equally: additional data collection could tell us what percentage of other experts will feel that such manipulations are acceptable.

This preliminary report also suggests implications for teaching. Even though only four subjects were interviewed, their concept images had some variety to them. One could say that there was a continuum of answers, from Johnny’s central thought that all differentials outside of integration were really the same and had analytic properties, to Kurtis’s assertion that all differentials were only part of a really good notation, with Sonya’s and Jackson’s views falling somewhere in the middle. Further research might further define spaces on this continuum or perhaps show a greater concentration of images at one or both ends. Whatever the dispersion of concept images on this continuum, the fact that such a dispersion exists perhaps begs the question of how the existence of different views of differentials held by textbooks and instructors might affect student learning.
References


