Angle Measure, Quantitative Reasoning, and Instructional Coherence: The Case of David

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This paper reports findings from a study that explored the effect of a secondary mathematics teacher’s level of attention to quantitative reasoning on the quality and coherence of his instruction of angle measure. I analyzed 37 videos of an experienced teacher’s instruction to characterize the extent to which he attended to supporting students in reasoning quantitatively, and to examine the consequences of this attention (or lack thereof) on the quality and coherence of the meanings the teacher’s instruction supported. My analysis revealed that the incoherencies in the teacher’s instruction were occasioned by his inattention to quantitative reasoning. This study therefore demonstrates that when teachers do not possess a disposition to attend to quantities and their relationships, the circumstances are ripe for instruction that emphasizes inconsistent, incoherent, and sometimes incompatible, mathematical meanings.

Key words: Mathematical Knowledge for Teaching, Quantitative Reasoning, Angle Measure, Radical Constructivism, Pre-Service Teacher Preparation

Introduction

Thompson (2013) argued that the quality of mathematics instruction in the United States suffers from “a systemic, cultural inattention to mathematical meaning and coherence” (p. 57). While a number of empirical studies demonstrate that Thompson’s accusation applies to a spectrum of mathematics courses and topics (e.g., Ma, 1999; Stigler & Hiebert, 1999), several researchers have noted that pre- and in-service teachers’ personal understandings of trigonometry, as well as their instruction, tend to be particularly lacking in coherence and conceptual meaning (Akkoc, 2008; Moore et al., in press; Tallman, 2015; Thompson, 2008; Thompson, Carlson, & Silverman, 2007). Others have observed that trigonometry is a notoriously difficult subject for students (Moore, 2012, 2014; Weber, 2005). Identifying the factors that contribute to widespread incoherent instruction of trigonometry is therefore a priority for improving students’ learning of the subject.

A growing body of research (e.g., Castillo-Garsow, 2010; Confrey & Smith, 1995; Ellis, 2007; Moore, 2012, 2014; Moore & Carlson, 2012; Oehrtman, Carlson, & Thompson, 2008; Thompson 1994) has identified quantitative reasoning (Smith & Thompson, 2007; Thompson, 1990, 2011) as a powerful way of thinking that supports students in constructing a meaningful understanding of a wide variety of mathematics concepts. Several researchers have noted that quantitative reasoning is especially foundational for supporting students’ conceptual learning of angle measure and trigonometric functions (Hertel & Cullen, 2011; Moore, 2012, 2014; Tallman, 2015; Thompson, 2008). However, the instructional consequences of teachers’ attention to quantitative reasoning (or lack thereof) are less frequently documented. By instructional consequences, I do not mean the understandings students construct while engaged in instructional experiences that emphasize (or fail to emphasize) quantitative reasoning. I am instead referring to the characteristics of the instruction itself, namely the nature of the understandings it promotes as well as its coherence.

Discerning how teachers’ attention to quantitative reasoning affects the quality and coherence of their trigonometry instruction has the potential to inform instructional and
curricular innovations that seek to improve the quality of this instruction, both in the United States and internationally. For this reason, I designed the present study to achieve such discernment. Specifically, I explored the effect of a secondary mathematics teacher’s level of attention to quantitative reasoning on the coherence of the meanings of angle measure his instruction supported.

Quantitative Reasoning

I leveraged Smith and Thompson’s (2007) and Thompson’s (1990, 2011) explicit formalizations of quantitative reasoning in the design of this study and in my analysis of its data. Quantitative reasoning is a characterization of the mental actions involved in conceptualizing situations in terms of quantities and quantitative relationships. A quantity is an attribute, or quality, of an object that admits a measurement process (Thompson, 1990). One has conceptualized a quantity when she has identified a particular quality of an object and has in mind a process by which she might assign a numerical value to this quality in an appropriate unit (Thompson, 1994). It is important to emphasize that quantities do not reside in objects or situations, but are instead constructed in the mind of an individual perceiving and interpreting an object or situation. Quantities are therefore conceptual entities (Thompson, 2011; Thompson et al., 2014).

Conceptualizing a quantity does not require one to assign a numerical value to a particular attribute of an object. Instead, it is sufficient to have a measurement process in mind and to have conceived, either implicitly or explicitly, an appropriate unit of measure. Quantification refers to the mental actions involved in conceptualizing an appropriate unit of measure as well as a measurement process, and results in an understanding of “what it means to measure a quantity, what one measures to do so, and what a measure means after getting one” (Thompson, 2011, p. 38). I emphasize that one need not measure an attribute of an object to have quantified it, but must have in mind a process by which she might do so (Thompson, 1994).

The quantities one might construct upon analyzing a situation are not limited to those whose numerical values are attainable from direct measurements. Defining a process by which one might measure a quantity often involves an operation on two or more previously defined quantities. In such situations, we say that the new quantity results from a quantitative operation—its conception involved an operation on other quantities. Quantitative operations result in a conception of a single quantity while also defining the relationship between the quantity produced and the quantities operated upon to produce it (Thompson, 1990, p. 12). It is important to draw attention to the distinction between a quantitative operation and a numerical or arithmetic operation. Arithmetic operations are used to calculate a quantity’s value whereas quantitative operations define the relationship between a new quantity and the quantities operated upon to conceive it (Thompson, 1990).

Methods

The sole participant for this study was an experienced secondary mathematics teacher, David, who taught Honors Algebra II at a large suburban high school in the Southwestern United States. David used the Pathways Algebra II (Carlson, O’Bryan, & Joyner, 2013) curriculum materials in this course. The Pathways Algebra II materials are organized into modules, each of which contains a number of investigations that students are expected to work on in small groups during class sessions.
I collected data throughout David’s instruction of Module 8 of the *Pathways Algebra II* curriculum. This module focuses on a variety of ideas related to trigonometric functions including angle measure, the output quantities and graphical representations of various periodic functions, periodic function transformations, and inverse trigonometric functions. In this paper, I present only the results of my analysis of David’s instruction of angle measure. I do not discuss in detail my analysis of David’s instruction of other topics addressed in Module 8 of the *Pathways Algebra II* curriculum since the conclusions I drew therefrom are consistent with those I present below.

David taught two sections of Honors Algebra II every weekday during the spring semester of 2014. I video recorded both classroom sessions over a seven-and-a-half-week period of this semester, which resulted in 37 videos of David’s instruction. The only class sessions that I did not videotape were those during which students were testing or those in which David was teaching content unrelated to trigonometric functions. In addition to the video recordings of David’s instruction, I generated field notes during the class sessions that focused on characterizing the extent to which David supported students in reasoning quantitatively, and on documenting the mathematical meanings David’s instruction promoted.

The procedures I used to analyze the video data are consistent with Strauss and Corbin’s (1990) and Corbin and Strauss’s (2008) grounded theory approach. I began my analysis of these videos by making an initial pass of open coding during which I identified instances that David conveyed some way of understanding. I coded these occasions for the specific category of understanding David communicated. I then made a pass of axial coding in which I verified and refined my initial codes. After having coded the 37 videos of David’s classroom teaching, I produced a 57-page document entitled, “Post-Analysis Memos” wherein I summarized each coded instance of the videos and included selective transcriptions of what appeared to be particularly revealing moments of David’s instruction. These memos also focused on characterizing for each coded instance the extent to which David supported his students in reasoning quantitatively. In particular, I documented the degree to which David’s instruction supported students in: (1) identifying quantities, (2) attending to units of measure, (3) constructing quantitative relationships, and (4) interpreting mathematical symbols and expressions as representing the values of quantities. I carefully read through these memos and organized the coded segments of video into themes. I then examined the data within each theme and characterized the extent to which the quality and coherence of the meanings David’s instruction promoted was facilitated/impeded by his level of attention to supporting students in reasoning quantitatively.

**Results**

Meaningfully assigning numerical values to the “openness” of an angle requires that one has identified a quantity to measure and has specified a unit with which to measure it. David’s instruction was often inconsistent with regard to the quantity one measures when assigning numerical values to the “openness” of an angle. On some occasions David supported students in conceptualizing angle measure as the length of an arc the angle subtends, while on other occasions he explained that measuring an angle involves determining the fraction of the circle’s circumference subtended by the angle. These meanings are not the same. Understanding the fraction of a circle’s circumference that an angle subtends as a measure of subtended arc length involves conceptualizing the circle’s circumference as a unit of measure for the length of the subtended arc. Specifically, one must recognize that the resulting fraction represents a
multiplicative comparison of the quantity being measured (subtended arc length) and the unit of measure (circumference). For example, to say an angle subtends $\frac{59}{360}$ths of the circumference of a circle centered at its vertex is to say that the length of the subtended arc has a measure of $\frac{59}{360}$ in units of one circumference. The two meanings of angle measure David conveyed were distinct since he did not support students in conceptualizing the circumference of the circle centered at the angle’s vertex as a unit of measure for the length of the subtended arc. Due to space limitations, the following paragraphs illustrate only two occasions in which David’s instruction supported inconsistent meanings of angle measure (from my perspective). I emphasize that the events discussed here are representative of several instances from David’s teaching in which he promoted discrepant meanings.

Lessons 1 and 2
David began the first lesson of Module 8 by asking a student to draw two angles on the whiteboard. The student drew one angle above the other. David then explained that the measure of the angle on top is larger than the measure of the angle on bottom because, if one were to construct two circles of equal radii respectively centered at the vertex of each angle, the angle on top would subtend an arc that is longer than the arc subtended by the angle on bottom. Immediately following this explanation, David asked the question in Line 1 of Excerpt 1.

Excerpt 1

1. David: When we measure an angle what are we really measuring? I mean it’s not like we’re measuring a length, right? How would we describe the thing that I’m measuring when I just look at these two angles? …

2. Student: The openness of the angle.

3. David: Yeah. Which is weird. How do you measure openness? … I’m not measuring length. … We have to think about what we are actually measuring.

While David previously compared the openness of two angles by attending to the respective arc lengths these angles subtend, he claimed in Lines 1 and 3 of Excerpt 1 that quantifying the openness of an angle does not involve measuring a length. Following the dialogue in Excerpt 1, David explained that two angles have the same measure if “the length of the [subtended] arc is the same, as long as I made the circle have the same radius and it was centered at the vertex.” David therefore supported contradictory meanings of angle measure during the first lesson; he pronounced that measuring an angle is not a process of measuring a length and then proceeded to compare the openness of two angles, as well as define what it means for two angles to have the same measure, by attending to the arc lengths the two angles respectively subtend. In other words, when speaking of angle measure David did not consistently reference the same quantity being measured.1

A few minutes after David’s remark in Line 3 of Excerpt 1, he projected the image displayed in Figure 1 on the whiteboard and asked his students the question in Line 1 of Excerpt 2.

1 Understanding angle measure quantitatively involves conceptualizing a specific attribute to measure as well as identifying an appropriate unit with which to measure it. When measuring an angle, the attribute one measures is the length of the arc the angle subtends. This subtended arc length must be measured in units that are proportionally related to the circumference of the circle that contains the subtended arc so as to make the size of this circle inconsequential to the measure of the angle. It is important to note that this condition on the unit of measure does not change the quantity being measured: subtended arc length.
Excerpt 2

1. **David:** So the angle subtends $1/8$ of the circumference of the circle. (*Pause*) Now do units matter here? … Why do units not matter here?
2. **Student:** ‘Cause you’re using a proportion.
3. **David:** Why does that matter? …
4. **Student:** Because even though you’re making the radius larger you’re also making the whole circle larger.
5. **David:** So what happens when you do your proportion? Think in science class. (*Long pause*) ‘Cause we’re comparing it to our circumference, right? We’re comparing arc length to circumference? What would happen to the units then? (*Long pause*) So let’s just say for the sake of argument $1/8$ could be a circumference of, uh, a circumference of 16, that would mean that the arc length would be two, if it’s an eighth. So two inches divided by 16 inches is?
6. **Student:** One-eighth.
7. **David:** One-eighth. What are the units now? (*Long pause*) What happens when you put—and again think in terms of science class—what happens when you put two inches divided by 16 inches (*writes “2in/16in”), your science teacher would say that’s $1/8$th. What are the units?
8. **Student:** It doesn’t matter.
9. **David:** It does matter. What are the units?
10. **Student:** Inches.
11. **David:** Inches divided by inches give you inches?
12. **Student:** No.
13. **David:** What does it give you?
14. **Student:** One-eighth.
15. **David:** What are the units?
16. **Student:** It doesn’t have units.
17. **David:** It doesn’t have units? Why not?
18. **Student:** Because the inches cancel.
19. **David:** ‘Cause inches cancel inches! … ‘Cause I’m not just measuring arc length. What am I measuring? I’m measuring arc length and comparing it to what?
20. **Student:** Circumference.
21. **David:** Circumference! How am I comparing them?
22. **Student:** By length.
23. **David:** By length? What operation is going on here? Am I subtracting the circumference? (*Pause*) It’s division! We’re creating a ratio! Then do the units matter?
24. **Student:** No. …
What happens when we do the ratio? The units stop mattering, right? Because the units end up canceling. We’re interested in the ratio. We’re not interested in the units from the ratio because the units are going to reduce.

After acknowledging in Line 1 of Excerpt 2 that the angle in Figure 1 “subtends 1/8th of the circumference of the circle,” David exclaimed that the measure of the angle is a value without units. In particular, David explained that if one measured the subtended arc length and circumference in inches, the ratio of these quantities is unit-less because the inches “cancel” as a result of the division. Moreover, by claiming, “I’m not just measuring arc length. What am I measuring? I’m measuring arc length and comparing it to what? … Circumference!” David did not support his students in seeing the ratio of subtended arc length to circumference as the length of the subtended arc measured in units of the circumference. Generally speaking, David’s statements and questions in Excerpt 2 did not provide students with an opportunity to interpret the ratio of subtended arc length to circumference as a quantitative operation but rather as an arithmetic operation; that is, he did not communicate the division of these quantities as a measure of subtended arc length in units of circumference but simply as the ratio of two lengths. Such an emphasis is necessary if one is to support students in conceptualizing angle measure quantitatively (i.e., as a measure of some attribute in some unit). Therefore, while David’s instruction during the first lesson overtly emphasized angle measure as the fraction of the circle’s circumference subtended by the angle—and implicitly conveyed angle measure as the length of the subtended arc—he did not encourage students to see the former meaning as an application of the latter by failing to support them in conceptualizing the ratio of subtended arc length to circumference as a quantity that represents the length of the subtended arc measured in units of the circumference. In fact, David suggested that these meanings were incompatible by continually asserting that the process of measuring an angle is not one of measuring length.

David’s instruction of angle measure did not consistently support students in conceptualizing the quantity one measures when assigning numerical values to the openness of an angle. On some occasions David conveyed that the process of measuring an angle is one of determining the length of the arc an angle subtends, while on other occasions he explained that measuring an angle involves determining the fraction of a circle’s circumference subtended by the angle. While in certain circumstances one of these ways of understanding might be more natural than the other, David did not support his students in making the connection between these ways of understanding. In other words, David did not provide opportunities for students to see these meanings as two instantiations of the same quantification process (measuring the length of the subtended arc in units proportional to the circumference of the circle containing the subtended arc) because he did not support students in conceptualizing the circumference of the circle centered at the angle’s vertex as a unit of measure for the length of the subtended arc.

Summary of Lessons 1-9

While David’s instruction during the first lesson emphasized the understanding of angle measure as the fraction of the circle’s circumference subtended by the angle, he also discussed angle measure as the length of the arc an angle subtends, but then suggested that these two meanings are incompatible by explaining that the process of measuring an angle is not one of measuring a length. The majority of David’s instruction during Lessons 2 and 3 emphasized angle measure as the fraction of the circle’s circumference subtended by the angle. Specifically, David’s teaching promoted the mental imagery of imagining the circumference of the circle centered at the vertex of the angle being split into a number of equal pieces, and then attending to the ratio of the number of these pieces subtended by the angle to the number of these pieces.
contained in the circumference of the circle. However, on several occasions during Lessons 4-9, David discussed angle measure as the length of the subtended arc measured in units of the radius of the circle containing the subtended arc. Moreover, David’s instruction throughout Lessons 4-9 was often contradictory in that he encouraged meanings that on other occasions he did not accept and devalued meanings that he elsewhere endorsed.

Discussion

The results of this study demonstrate that a secondary mathematics teacher’s (David) inattention to quantitative reasoning contributed to his conveying incoherent meanings of angle measure and trigonometric functions. In particular, by not maintaining a consistent emphasis on supporting students in: (1) identifying quantities, (2) attending to units of measure, (3) constructing quantitative relationships, (4) interpreting mathematical symbols and expressions as representing the values of quantities, and (5) performing quantitative—rather than arithmetic—operations, the ways of understanding David’s instruction promoted were often inconsistent and even incompatible. Moreover, the meanings David supported varied by context because they were not consistently governed by, or in the service of promoting, a particular way reasoning. These findings suggest implications for secondary mathematics teacher preparation.

While the well-documented affordances of quantitative reasoning on students’ conceptual mathematics learning are enough to justify its emphasis in pre- and in-service teacher education, the results of the present study provide an additional incentive for mathematics educators to engage teachers in experiences that advance their ability to reason quantitatively, as well as support them in leveraging quantitative reasoning in their teaching of specific concepts. Specifically, the results of this study suggest that teacher educators should design instructional experiences that allow pre- and in-service teachers to develop the disposition to support students’ identification of quantities and quantitative relationships, and their interpretation of mathematical symbols and expressions as representations of a measure of an attribute of some object in some unit, particularly in the difficult context of trigonometry. Equipped with such a disposition, teachers’ instructional actions may consistently be in the service of leveraging a powerful way of reasoning to support students’ learning of various mathematical ideas while simultaneously promoting the way of reasoning itself. The results of the present study demonstrate that when teachers do not possess a disposition to attend to quantitative reasoning, the circumstances are ripe for instruction that emphasizes inconsistent, incoherent, and sometimes incompatible, mathematical meanings.

References


