Students’ Understanding of Vectors and Cross Products: 
Results from a Series of Visualization Tasks

Monica VanDieren 
Robert Morris University

Deborah Moore-Russo 
University at Buffalo, SUNY

Jillian Wilsey 
Niagara County Community College

Paul Seeburger 
Monroe Community College

Previous studies have explored student understanding of vectors in physics, engineering, or linear algebra settings, but there has been scant research on student understanding of vectors in a multivariable calculus context. In this study, we begin to explore how students think about vectors and cross products by analyzing student responses to open-ended questions from an online, conceptually-oriented multivariable calculus cross product activity. We identify several themes consistent with previous research on physics students including confusion between the cross product and its magnitude as well as difficulty identifying or communicating the direction of the cross product vector. This preliminary research begins to develop categories that could outline a conceptual model of student understanding of vectors and cross product. The analysis also informs several recommendations for improving the cross product activity.

Key words: [vectors, cross products, student understanding, visualization]

In mathematics, engineering, and physics, vectors play a foundational role. Vectors are used extensively throughout physics and engineering within mechanics topics such as force, velocity, and acceleration, and in mathematics coursework vectors appear in multivariable calculus, geometry, linear algebra, and differential equations. While basic vector concepts, representations, and operations with vectors are presented in both high school and preliminary college mathematics, students often will not be introduced to vector dot and cross product operations until they undertake college-level calculus coursework. Despite the regular occurrence throughout the curriculum, students continue to have significant conceptual difficulties with vector concepts and manipulations.

One approach conjectured to improve student understanding of three-dimensional topics is to improve students’ visualization skills using computer exercises (Sorby & Baartmans, 2000). CalcPlot3D is an online, freely available 3D-graphing applet that allows students to visualize and manipulate concepts including vectors, vector fields, parametric curves, surfaces, and gradients. In addition to the graphing calculator feature, CalcPlot3D offers discovery-learning activities for students to explore multivariable calculus concepts (Seeburger, 2016). This study analyzes student responses to Seeburger’s cross product activity.

The purpose of this study is to investigate students’ transitional conceptual understanding of vectors, and more specifically the cross product of vectors, as they work through an exploration activity with embedded questions while using the CalcPlot3D visualization tool. This research is partially supported by the National Science Foundation under Grant Numbers 1524968, 1523786, 155216.
Background Literature

There is extensive literature that explores students’ misconceptions when confronted with problems involving force and motion (i.e., mechanics), both of which are represented by vectors (Aguirre & Rankin, 1989; Barniol, Zavala, & Hinojosa, 2013; Flores, Kanim, & Kautz, 2003; Hestenes & Wells 1992; Hestenes, Wells, & Swackhamer, 1992; Miller-Young, 2013). Both force and motion utilize vector concepts; however, the students’ misconceptions regarding vector concepts, properties, and fluency in vector operations are not explored directly. Rather, these concepts are embedded within the application. For instance, Hestenes, Wells, and Swackhamer (1992) utilize a Force Concept Inventory to assess student understanding of Newtonian physics, but the inventory does not directly assess students’ understanding of vectors.

Others (Barniol & Zavala, 2014; Knight, 1995; Nguyen & Metzler, 2003; Van Deventer & Wittmann, 2007; Wang & Sayre, 2010; Zavala & Barniol, 2010) provide more explicit consideration of students’ understanding of vector concepts, representations, and operations outside of a kinematic or mechanics context. Knight (1995) found that approximately 40% of students in an introductory calculus-based physics course had no idea what a vector was. About 50% of the students could add vectors correctly; however, none of the students were able to evaluate a vector cross product. Barniol and Zavala (2014), using their Test of Understanding of Vectors (TUV), examined the knowledge of university students who had completed an introductory calculus-based physics course. The TUV contains non-contextual multiple-choice problems covering vector properties and basic vector operations. Three of these problems address the cross product. Two are computations and the third asks the students to select an appropriate geometric interpretation of the cross product from a list of options. The percentage of students who could correctly answer this problem was 57% (Barniol & Zavala, 2014).

Research on student understanding of vectors in college-level mathematics courses tends to focus on the transitional proof courses such as linear algebra and geometry. For instance, Stewart and Thomas (2009) developed a framework for vectors in linear algebra that combines an action-process-object schema (Dubinsky, 1991) for vectors with Tall’s (2004) categorization of three mathematical ways of thinking: embodied world, symbolic world, and formal world. Kwon (2013) presents a new framework for conceptualizing vectors in college geometry that identifies three representations of a vector: vector as a translation; vector as a point and point as a vector; and geometrical vector sum.

Theoretical Framework

While many studies highlight student misconceptions, this study focuses on students’ transitional conceptions, since understanding is not necessarily static. Transitional conceptions relate to students’ current notions of a concept that are cued by the task at hand and that may include what some would call misconceptions. Transitional conceptions are not fully integrated in a coherent manner, and hence tend to be in flux. Yet, they do result from a sense-making activity even though they may only address some (but not all) aspects of the concept and may be productive in some (but not all) contexts. Studying transitional conceptions can potentially lead to a better or more accurate view, perhaps even to a conceptual model, of student understanding of the concept (Moschkovich, 1999). To understand students’ meaning-making processes, it is imperative that instructors consider the transitional conceptions that occur when students are
engaged in learning new concepts (Wolbert, Moore-Russo, & Son, 2016). Many have begun to consider college students’ transitional concepts (Chiu, Kessel, Moschkovich, & Muñoz-Nuñez, 2001; Cho & Moore-Russo, 2014; Nagle, Casey, & Moore-Russo, 2015; Wolbert, Moore-Russo, & Son, 2016) in various areas of post-secondary mathematics. However, there is little research on transitional conceptions specific to vectors in multivariable calculus classes. This study aims to add to the existing body of knowledge, focusing, in particular, on students’ transitional conceptual understanding of the vector cross product.

Conceptual understanding encompasses both “what is known (knowledge of concepts)...[and] the way that concepts can be known (e.g. deeply and with rich connections)” (Star, 2005, p. 408). It can be considered as “a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information” (Hiebert & LeFevre, 1986, p. 4). When concepts are first learned, students’ understanding can be fragmented and lack either organization or connections to related concepts (Schneider & Stern, 2009).

Both intra- and inter-connections are possible for a concept. Tall and Vinner (1981) described an individual’s concept image as the entire cognitive structure related to a particular concept. Past research has considered the many “conceptualizations,” “notions,” or “connected components” associated with a particular concept in mathematics (e.g., work on slope by Moore-Russo, Conner, & Rugg, 2011; Nagle & Moore-Russo, 2013); these are the intra-connections for a single concept. There are also inter-connections between different topics (e.g., Zandieh and Knapp’s (2006) work that inter-relates rate, limit, and function to look at how students come to understand derivatives); however, this paper focuses only on intra-connections.

Understanding is dynamic, and students’ conceptions regarding a topic are transitional. As students advance in mathematics, there are instructional expectations that they will develop internal networks that are rich in relationships where they are able to move flexibly among representations and notions of the concept as they advance in the learning of a topic. Assessment to confirm that students are making such intra-connections to form robust, flexible concept images is important since conceptual understanding, in combination with procedural fluency, is necessary for success in mathematics (Hiebert & Carpenter, 1992).

Methods

This study is situated within a larger research program to determine if student understanding of cross products can be enhanced through visual explorations. Before being able to assess the impact of visual explorations on student understanding, it is necessary to build a model of student understanding of the cross product. As a first step to this goal, we sought to begin to characterize students’ transitional conceptual understanding of cross products by examining their responses to four open-ended questions on a conceptually-oriented cross product assignment.

Subjects and Setting

The data analyzed were from electronic responses of 434 college-level multivariable calculus students to four open-ended questions from an online assignment. The data was collected over four years from students from community colleges, four-year private colleges, and four-year public colleges. Each student completed a pre-test, exploration assignment, and post-test.

The exploration assignment consisted of 10 open-ended and 2 multiple choice questions about vectors and cross products. The students were directed to a visual applet that contained two
vectors (one red and one blue) along with their cross product. The red and blue vectors were graphed with initial points situated at the origin in the \(xy\)-plane. Students could manipulate the length and direction of the two vectors on the \(xy\)-plane, and based on the students' input, the applet automatically redrew the cross product, computed the magnitude of the cross product, and indicated the angle between the two given vectors.

Previous research on the pre- and post-test multiple choice questions indicate some knowledge gain on the relationships between the angle between two vectors or between the length of two vectors and their cross product through the use of the exploration (Seeburger, 2009). Here we examine student responses to the first four open-ended questions embedded in the pre-existing exploration to gain a better view of how students understand cross product. The remaining questions will be examined in a follow-up study. The questions examined here are:

Q1. What is the geometric relationship between the cross product vector and the two vectors that form it? (Hint: This is NOT a formula.)
Q2. How is the cross product vector geometrically related to the two vectors that form it? (Hint: This is NOT a formula.)
Q3. For vectors of fixed length, but varying the direction of one of the vectors, when is the magnitude of the cross product at a maximum?
Q4. For vectors of fixed length, but varying the direction of one of the vectors, when is the magnitude of the cross product at a minimum?

Analytical Method

The items were examined for emerging themes through a general inductive analysis. According to this method, the researcher does not begin with a preconceived structure but allows categories to emerge from the data. The researchers utilize categories to make sense of what is observed (Thomas, 2006). To identify emerging categories, one member (the first author) of the research team began the task of reading all responses to the four items to note what was observed. These emerging categories were shared with another member of the research team (the fourth author) for general consensus. A third member of the research team (the second author) then read all the responses and tried to see if the initial categories could be collapsed into unifying topics. She shared her findings with the third author until consensus was reached; both agreed on all the identified categories, but refinements were made to the topic descriptions. The list of categories was then shared and discussed with the first author, who also agreed with the identified categories and then helped further refine the topic descriptions. All, but the miscellaneous incorrect category and the blank category, were coded on three levels: as not being present (0), being present but with a developing or transitional understanding (-), or being present with a correct or accurate (+), but perhaps not complete, understanding. Finally the first and second authors coded all the items using the category descriptions in Table 1.

After coding was completed, interrater reliability statistics for each code were computed with ReCal (Freelon, 2013). With the exception of the miscellaneous category, all codes resulted in Krippendorff’s alpha above 0.80. The first and second authors then came to a consensus on the few instances in which they were in disagreement.
<table>
<thead>
<tr>
<th>Category and Codes</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angle between two vectors</td>
<td>V+ angle must be between 0 and 180 degrees&lt;br&gt;V- angle between two vectors can be negative or greater than 180 degrees</td>
</tr>
<tr>
<td>Orthogonality</td>
<td>O+ cross product vector is orthogonal/perpendicular/90 degrees from the two vectors that form it&lt;br&gt;O- developing, but not correct/precise, statement about orthogonality</td>
</tr>
<tr>
<td>Right-hand Rule</td>
<td>R+ Mention of the right hand rule; Statement that correctly addresses the need to attend to orientation of one vector relative to another&lt;br&gt;R- developing, but not correct/precise, statement about the right hand rule</td>
</tr>
<tr>
<td>Parallelogram</td>
<td>P+ magnitude of cross product is area of parallelogram formed by two vectors&lt;br&gt;P- developing, but not precise, statement about parallelograms or area (e.g. “cross product forms a parallelogram.”)</td>
</tr>
<tr>
<td>Formula for magnitude of the cross product</td>
<td>F+ magnitude of cross product =</td>
</tr>
<tr>
<td>Angle impact on magnitude of the cross product</td>
<td>A+ correct statement describing how the angle between vectors influences the magnitude of the cross product&lt;br&gt;A- incorrect or vague statement about how changing the angle between two vectors will affect the length of the cross product (e.g. “cross product depends on the angle between the two vectors”)</td>
</tr>
<tr>
<td>Length impact on magnitude of the cross product</td>
<td>L+ correct statement describing how the length of the two vectors influences the magnitude of the cross product&lt;br&gt;L- incorrect or vague statement about how changing the length of one of the vectors will affect the length of the cross product</td>
</tr>
<tr>
<td>Other incorrect</td>
<td>I incorrect statement involving x and y coordinates; vector addition; statement involving quadrants or planes; other nonsensical statements</td>
</tr>
<tr>
<td>Blank</td>
<td>B no response; “I don’t know”</td>
</tr>
</tbody>
</table>

Table 2

| Codes | V+ | V- | O+ | O- | R+ | R- | P+ | P- | F+ | F- | A+ | A- | L+ | L- | I | B |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|---|---|
| Q1    | 217| 8  | 4  | 2  | 23 | 52 | 4  | 19 | 6  | 8  | 5  | 29 | 81 |
| Q2    | 7  | 256| 10 | 10 | 4  | 30 | 4  | 15 | 25 | 17 | 1  | 6  | 14 | 56 |
| Q3    | 1  | 29 | 2  | 3  | 371| 17 | 7  | 1  | 4  | 44 |
| Q4    | 29 | 5  | 4  | 348| 37 | 6  | 1  | 7  | 44 |
| Tot.  | 1  | 65 | 473| 18 | 13 | 53 | 92 | 8  | 41 | 750| 79 | 7  | 19 | 51 | 225 |
Results

In Table 2 we summarize the results of the coding by question for the 434 students. While some of the codes were rarely assigned (e.g. L+, L-), the complete data set contains responses to other questions in which these themes are more prevalent. The authors plan on applying the methods tested in this initial analysis to the remaining questions in the exploration in the next phase of research.

Overall students evidenced some understanding of the cross product. The percentage of students providing a correct (+) response which may, or may not, be complete, for either Q1 or Q2 was 75% and for Q3 or Q4 was 88%. Note that some of these students may have also indicated additional incorrect or transitional understanding (-) of another aspect of the cross product in their responses as well.

Although the first two questions are nearly identical, some students answered them differently. Another theme we noted in Q1 and Q2 answers was that only 4% of all students referred to the right-hand rule or the orientation of the cross product to the two vectors that formed it (in either a correct (+) or transitional (-) way). The students who did describe the orientation of the cross product to the two vectors that form it appear to be recalling previous knowledge since the right-hand rule is not explicitly referred to in the exploration until later on. Nearly all of these students only quote it by name: “These vectors also form what is called the Right Hand Rule.” Only a few students made an attempt at describing the right-hand rule in their own words. For example:

The direction of the cross product is found by using the right hand rule which involves placing your thumb on the first vector of the cross product and your forefinger on the second vector and seeing where your middle finger is pointing. The direction it is pointing is the same as the direction of the cross product.

Furthermore, when considering the relationship between the two given vectors and their cross product, students more often only describe how the angle between the two given vectors impacts the length of the cross product, but do not consider how the length of the two given vectors would affect the magnitude of the cross product.

Discussion

Although concentrated to the analysis of four open-ended questions concerning the cross product, this study reveals difficulty in student understanding of cross products consistent with the literature. In particular, students have difficulty with the right-hand rule, tend to rely on formulas, and confuse the cross product with its magnitude. Further analysis of the remaining cross product exploration questions along with this preliminary study will inform recommendations for future versions of the CalcPlot3D concept exploration and provide insight into the intra-connections of vector cross products that students are and are not understanding.

While 65% of students correctly stated that the cross product was orthogonal to the two vectors that form it in responses to the first two questions, only 4% of students made any reference to the right-hand rule, orientation, or noncommutativity of the cross product vector. Similar results are found in a physics context by Zavala and Barniol (2010) and by Scaife and
Heckler (2010). Note, however, that in their research the students were required to compute the cross product while in our study the cross product was provided to the students graphically. Barniol and Zavala found that 44% of the students were able to interpret the cross product (AxB) as a vector perpendicular to both A and B, although only 22% of the students identified the correct direction. In the context of magnetic force, Scaife and Heckler (2010) saw that in a series of four similar questions, 40% of students made a sign error for the cross product at least once. They conjecture this is due to confusion about the application of the right-hand rule and failure to recognize that the cross product is a noncommutative operation.

Another theme found in our data was that 5% of the responses to Q1 and Q2 in our study included some (either correct or incorrect) reference to a formula, even though both questions explicitly state “Hint: This is NOT a formula.” This supports research of Zavala and Barniol (2010) who found 9% of third semester physics students studied referred to a formula for the cross product in their open-ended responses to interpret the cross product. This additionally advances the notion, witnessed by Miller-Young (2103) in the context of an engineering class, that students often rely on memorized equations and procedures, even when instructed not to do so.

Zavala and Barniol’s (2010) analysis showed 32% of students did not distinguish the cross product magnitude from the cross product itself. In our data set, this same trend is present in many of the (P-) responses similar to “The area of the two vectors equals the cross product.” On the other hand 50% of students in our study focused on the direction of the cross product and did not describe its magnitude. This might suggest that students did not have a robust concept image, or they might not have seen the need to report on all connected components of the concept.

**Future Recommendations**

One of the limitations of the CalcPlot3D exploration is that the applet does not allow students to move the vectors off of the origin or off of the xy-plane. This may have led to incorrectly over-generalized responses; for example, “[The cross product] point[s] in the z axis direction.” Marton and Booth’s Variation Theory suggests that activities be structured to ensure students experience a diversity of examples (Lo, 2012). In this case students should experience vectors in a variety of orientations, not just those on the xy-plane.

The exploration provides students the opportunity to communicate and describe the geometric features of vectors and their cross product. Some students had difficulties accurately and precisely describing these features which may contribute to only a limited understanding of the relationships. For instance, in Q4, 86% of the students who described the angle between vectors as negative or larger than 180 degrees also demonstrated only a transitional understanding of the relationship between the angle of the vectors and the length of the cross product. Adding more examples of verbal descriptions of the geometric features of vectors may provide students with scaffolding to better communicate mathematics in open-ended responses.

The next step for the research team is to complete the analysis of all of the responses to the remaining exploration questions to gain more insight into what intra-connections of cross product are being made by the students and how the CalcPlot3D explorations can be improved to better address student difficulties.
References


