Considerations for Explicit and Reflective Teaching of the Roles of Proof

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In a previous study we sought to understand the classroom activities that provided students the opportunity to engage in the five roles of proof described by Michael de Villiers (1990). In conducting the analysis for that study, we noticed that students’ views of proof were sometimes not aligned with de Villiers’ views. This led us to the current investigation, where we explore alignment between undergraduate students’ views of the nature of proof and de Villiers’. We hypothesize that an explicit and reflective (ER) approach to instruction may be important if students are to learn about the nature of mathematics (in general) and the nature of proof (more specifically). We offer implications for both research and practice, with respect to the explicit and reflective instruction on roles of proof.

Key words: Roles of Proof, Nature of Mathematics, Nature of Proof, Transition-to-Proof

Researchers in mathematics education have made several efforts to understand the disciplinary practices of mathematicians (e.g., Burton, 1999; Nardi, 2008; Weber, 2008; Weber & Mejia-Ramos, 2011). As a result of such studies, the field benefits by gaining a deeper understanding of the nature of mathematical knowledge and inquiry. Proponents of situated learning theory (ourselves included) would argue that in order to learn mathematics, students must engage in authentic mathematical practices (Greeno, 1997; Lave & Wenger, 1991). From this perspective, understanding the work of experts in a field (such as mathematicians in mathematics) is an important aspect of instructor knowledge, allowing instructors to design learning environments that engage students in authentic disciplinary practices and thus aid in their learning of mathematics. But, do students need to go beyond engagement in legitimate mathematical practices within the classroom, and actually hold an understanding of the nature of mathematician’s practice and the nature of mathematical knowledge? It is widely acknowledged that most students know very little about what mathematicians do (Hersh, 1997). Yet little research has been conducted regarding students’ understanding of the nature of mathematics as a discipline (Jankvist, 2015). Perhaps an understanding of the nature of mathematical knowledge and inquiry may lead undergraduates to have a greater appreciation of mathematics or even lead to greater learning gains. But until systematic research is conducted into this area, these remain untested hypotheses.

Within science education, researchers have studied how students learn about the nature of scientific inquiry and scientific knowledge and the benefits of such knowledge (Lederman & Lederman, 2014). One of the main findings of that work is that engagement in authentic scientific practice alone is not sufficient for students (or teachers) to learn about the nature of science (Bell, Blair, Crawford, & Lederman, 2003). Although teachers often perceive that their students will implicitly learn the nature of science through engaging in scientific practice, research shows that students need explicit and reflective (ER) instruction on the nature of science in order to develop a sophisticated understanding (Bell et al., 2003). To teach the nature of science explicitly and reflectively means that students engage in authentic scientific practice, have that practice brought to their attention explicitly (e.g., by the instructor), and have the opportunity to reflect on the ideas that have been explicitly addressed (Lederman et al., 2014). Researchers in mathematics education sometimes claim that students have developed “desirable beliefs about the nature of mathematics” (Rasmussen & Kwon, 2007, p. 192) after participating in inquiry-oriented courses. However, these
claims about student beliefs are made in regards to what it means to do mathematics within the particular classroom settings, and generalizations are not made about what students may believe about the nature of mathematics as a discipline (e.g., Fukawa-Connelly, 2012; Yackel & Rasmussen, 2002).

We believe there may be an implicit assumption possessed by some scholars that if students participate in inquiry-oriented classrooms and engage in mathematical practices similar to those of research mathematicians, then the students may come away from such classes with informed conceptions of what it means to know and do mathematics in the discipline. The authors of this paper admit to being guilty of this assumption in the past. We believed that undergraduates’ understanding of the nature of proof in the discipline could be developed implicitly by engaging students in the five roles of proof described by de Villiers (1990). One of the learning objectives in our transition-to-proof course, taken from the course syllabus, was that students “Gain an appreciation of the many roles of proof and reasoning in the discipline of mathematics (e.g., verification, explanation, systemization, discovery, communication).” In a previous study, we identified the (classroom) activities that engaged students in those five roles of proof. Our implicit assumption was that by engaging in the five roles of proof, the students would gain a sophisticated understanding of those roles in the discipline. However, we found that even when students were engaged in a role of proof, this did not always lead to their understanding of the role in the broader discipline. Even after reading de Villiers’ (1990) paper on the roles of proof, their written summaries of these roles suggested they had naïve conceptions that did not align with de Villiers’. This leads us to the hypothesis, based on the findings in science education, that an explicit and reflective (ER) approach to instruction may be important if we want students to learn about the nature of mathematics (in general) and the nature of proof (more specifically).

In this preliminary report, we intend to explore the following questions:

- What do undergraduate students in a transition-to-proof course understand about the nature of proof in the discipline? What do they not understand?
- Which roles of proof are in need of further explicit and reflective (ER) instruction?

By exploring these questions, we hope to gain a better understanding of the considerations necessary when teaching the nature of proof in a more explicit and reflective (ER) manner.

**Conceptual Framework**

A key construct in our research is the notion of the nature of proof as it is experienced by research mathematicians. We conceptualize the nature of proof using de Villiers’ (1990) five roles of proof: verification, explanation, systematization, discovery, and communication. A mathematician engages in *verification* when a proof convinces the mathematician of the truth of a mathematical statement. The reason why the mathematical statement is true may be illuminated as a mathematician engages in the *explanation* role of proof. *Systematization* refers to proof’s role in organizing and creating a deductive system of axioms, definitions, and theorems. A mathematician engaged in *discovery* may deduce an unanticipated result during the completion of a proof. Proof also provides a means for *communication* among mathematicians as they transmit mathematical knowledge and negotiate meaning and validity. We frame our work from the following assumption: Students have a sophisticated understanding of the nature of proof if their ideas surrounding the roles of proof align with those of de Villiers (1990).
Methodology

The data for this study were taken from an undergraduate transition-to-proof course at a southeastern university in the United States. The instructor (also one of the authors on this paper) designed the course with an aim toward broadening students’ understanding of proof and providing experiences for students to engage with proof in ways similar to mathematicians in the discipline. There were thirteen students in the course: nine mathematics majors (seven of whom were prospective secondary mathematics teachers), and four mathematics minors. At the end of the semester the students took a two-part final exam. The second part (20% of the exam grade) of the final required students to read de Villiers (1990) paper, describe in their own words each of the five roles of proof, recall an instance during the course in which they engaged in one of the five roles (or describe an activity that might be used in the future for engaging students in such a role) and rank order their engagement in each role of proof throughout the semester.

The researchers used open process coding (Saldaña, 2009) to analyze the written descriptions of the 65 student recollections (five roles and thirteen students) of the instances in which they recalled being engaged in the roles of proof during the course. During subsequent analysis the researchers discussed the following questions:

1) How do students perceive that a certain activity engages them in a specific role of proof?
2) How do student perceptions align with the role of proof as articulated by de Villiers?
3) What are some implications for teaching such a course in the future?

When completing initial work related to the project, we focused on the first question. Here we concentrate our efforts on the second question related to comparing student perceptions to de Villiers’ perceptions. After completing the introductory analysis, the first author went back to the data and reviewed each students’ summary of the five roles of proof as well as their description of an activity which engaged them in the role of proof. He noted if the students’ ideas were aligned with de Villiers, identified evidence for this determination, and noted any implications. He then reviewed these notes and identified several preliminary discussion points regarding student understanding of the roles of proof and the nature of mathematical knowledge/inquiry.

Results and Discussion

Verification

When describing the verification role of proof, de Villiers’ (1990) challenged the naïve view that proof provides mathematicians with absolute certainty. While acknowledging that proof does indeed provide conviction, especially in the case of non-intuitive claims, lack of quasi-empirical falsification also plays a role in conviction. Our data suggest that reading de Villiers’ arguments was not enough for students in the course to understand this position. Several students summarized de Villiers’ description of verification using the language of “absolute certainty.” David wrote, “Verification by proofs is to show the absolute certainty of a mathematical computation by proving it in every case.” Similarly, Tina claimed,

The verification of a proof is most often a way of knowing what you already know. By that I mean the things that we are most sure are true the verification makes us absolutely certain. If mathematicians had not verified the proofs of some of the greats from ancient times we still would not know that some of them were incorrect.

Students such as Tina and David may benefit from classroom discussions regarding the nature of proof and absolute certainty. Perhaps reading and discussing relevant philosophy of mathematics
(Hersh, 1993) or mathematics education literature (Weber & Mejia-Ramos, 2015) may prove useful in challenging students’ absolutist conceptions of proof.

**Explanation**

We have noticed that the term “explanation” may be confusing for students. While the occasional student understood this role, many students interpreted explanation in a colloquial sense, e.g. explaining why one step follows from another within a proof. Jared claimed,

Explanation is the reasoning behind the proof. It shows how steps are logically taken to get to the conclusion of the proof and why we take them. Overall, it is saying why we can go from the statement to the conclusion.

Similarly, Stephanie wrote,

Proofs may provide more or less explanation and still be a valid proof. Explanation is just how much detail the writer of the proof goes into. If your audience doesn’t know as much about mathematics you may want to give a full explanation of your proof.

Perhaps it would be beneficial for an instructor to specifically draw attention to how proving can provide insight or understanding. Hersh (1993) claimed that in teaching, the primary role of proof is for explanation. However, without explicit and reflective instruction, students ultimately may fail to understand that proofs serve this important function. Instructors may ask students to compare and contrast proofs in regards to the insight or understanding they provide.

**Systematization**

In general, the students in the course under study did not understand the systematization role, that proof may play a role in the organization of a theory. They viewed systematization as the use of established theorems, axioms, and definitions within a proof. Jeb wrote,

Systemization is taking several smaller true statements and arranging them into one large true statement that is the proof. It is organizing or ordering the smaller sections of a proof to flow in such a way that they look like a single statement.

De Villiers (1990) noted that the systematization role is only understandable at an advanced stage of mathematics. It is the first author’s opinion that in a course in which systematization has already been conducted by the organization of course materials, it will be difficult for students to understand systematization as a role of proof. Perhaps in a course in which the organization of materials is not predetermined (e.g. Fawcett, 1938), students may better understand this role.

**Discovery**

De Villiers (1990), in describing the discovery role of proof, described how mathematicians may prove an unexpected result when they realize, through deduction, that a proof generalizes from a specific case to a larger class of mathematical objects. Students in the course recalled times when they experienced discovery in this manner, but also wrote about additional ways they came to discovery during the course. For instance, Millie recalled a problem that asked her to “Prove, or disprove and salvage” a given mathematical statement:

I decided [for all integers a, b, and c, if bc is divisible by a, then either b is divisible by a or c is divisible by a] was a false statement, and I provided a counter example. Then, I made a conjecture that the converse was true: for all integers a, b, and c, if either b is divisible by a or c is divisible by a, then bc is divisible by a. I proved my conjecture, and that conjecture was my own personal discovery.
Although not aligned with de Villiers’ description of the discovery role of proof, we believe that Millie’s learning experience was valuable. For instance, Susan explicitly mentions how classroom discoveries led her to develop a new conception of the nature of mathematics.

Discovery came to me when working on a truth table to test possible outcomes of a proof. This truth table led to “Susan’s conjecture.” I also think this class, in general, led to a broader “discovery” that mathematics is a living, changing, developing thing. Unlike my former perspective that it had all been discovered many years ago and we are just reviewing and learning those truths.

We contend, that although these two students did not describe an instance in which they discovered a new result through deduction (as de Villiers described), their classroom discoveries were valuable. It is important that students come to realize that mathematics is a dynamic field in which new discoveries are made rather than a static body of knowledge. Explicit and reflective instruction related to the discovery role of proof should enable students to be aware of the variety of ways discovery may occur in mathematics (in addition to discovery as de Villiers described).

**Communication**

De Villiers describes the communication role of proof as being related to the negotiation of meaning and validity amongst mathematicians. We are encouraged that students seemed to understand this role. Krissy wrote,

Our inability to come to a consensus among three people when evaluating a particular argument also demonstrated how difficult it might be for the global community of mathematicians to achieve agreement when it comes to proof style and validity.

The instructor designed several course activities with the goal of helping students understand the social nature of proof. For instance, students designed a course rubric that outlined what makes a valid mathematical argument and were constantly asked to critique the arguments of others. The instructor frequently made the social nature of proof an explicit part of classroom discussions and we believe this contributed to students’ understanding of the communication role of proof.

**Conclusions**

Reading de Villiers’ (1990) article may be a first step towards explicit and reflective (ER) instruction on the nature of proof, but it is not enough. The students submitted their reflections as part of a final assignment, and there was no subsequent class discussion related to the assignment. Our findings may have been different if students had the opportunity for discussion. We judge that if any role of proof was discussed explicitly and reflectively most often in class, it was the communication role. Students seemed to possess a clear understanding of how validity is negotiated within the community of mathematicians through proof.

In moving forward with this work, we would like to gain feedback from RUME conference participants on the following questions:

1. How important is it that students understand the nature of mathematics (in general) and the nature of proof (more specifically)? Why?
2. How can what we know from science education research (about explicit and reflective teaching of the nature of a discipline) transfer to mathematics education?
References


