Instructor-Generated Concepts Framework for Elementary Algebra in the College Context

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The long-term aim of this study is to develop a conceptual framework outlining the concepts necessary for college students to be able to successfully complete fundamental tasks of elementary algebra. This paper is a preliminary report of one part of this research, which focuses on instructor perceptions of what concepts are fundamental to successful completion of elementary algebra tasks. The framework presented here is the result of an action research project conducted by five college instructors in the U.S. who teach elementary algebra.

Keywords: Elementary algebra, conceptual understanding, algebra concepts, tertiary education.

Elementary algebra and other developmental courses have consistently been shown to be barriers to student degree progress and completion in the U.S.. There is evidence that only as few as 20% of students who are placed into developmental mathematics ever successfully complete a credit-bearing math course (see e.g. (Bailey, Jeong, & Cho, 2010). At the same time, elementary algebra has higher enrollments than any other mathematics courses at U.S. community colleges (Blair, Kirkman, & Maxwell, 2010).

Significant research has been done in the primary and secondary context to explore which types of student thinking lead to more successful or less successful outcomes in student algebraic problem-solving, but little research has been conducted with students enrolled in elementary algebra courses in the tertiary context, despite the fact that there is significant evidence to suggest that mathematics learning is likely somewhat different in this context (Mesa, Wladis, & Watkins, 2014). One approach to investigate this setting is to conduct participatory action research in the tertiary context in order to explore how instructor experiences, including cyclical investigations of their own practice, can shed light on some ways that tertiary students learn elementary algebra concepts, and on which types of student understandings are important for successful completion of elementary algebra tasks.

Conceptualizations of algebra and fundamental algebraic concepts

There are a number of different conceptualizations of algebra that have been explored in the research literature. Usiskin (1988) laid out four conceptualizations of algebra: generalized arithmetic; the set of all procedures used for solving certain types of problems; a study of relationships among quantities; and a study of structure. Kaput (1995) in contrast identified five conceptualizations of algebra, the first four of which mirror somewhat closely those of Usiskin: generalization and formalization; syntactically-guided manipulations; the study of functions, relations, and joint variation; the study of structure; and a modeling language.

A number of different important algebraic concepts have been studied previously, typically in the primary or secondary context. A complete review of the literature is not possible due to space constraints, but we outline here briefly some of the major categories of research on algebra that are relevant to elementary algebra in the tertiary context and cite one or two key references for each:

- Variables and symbolic representation (Dubinsky, 1991; Kuchemann, 1978; Sfard, 1991)

- Functions and covariation (Blanton & Kaput, 2005; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Goldenberg, Lewis, & O'Keefe, 1992)
- Equivalence (Kieran, Boileau, Tanguay, & Drijvers, 2013; Knuth, Stephens, McNeil, & Alibali, 2006)
- Algebraic structure sense (Hoch, 2003; Hoch & Dreyfus, 2006; Linchevski & Livneh, 1999)

For an excellent and systematic review of historical developments in the conceptualization, teaching, and learning of algebra, see (Kieran, 2007). Some researchers have developed frameworks for organizing algebra as a subject, typically in the primary and secondary context (see e.g. (Nathan & Koedinger, 2000; Sfard & Linchevski, 1994). In addition, various national standards regarding the teaching of algebra exist, such as the Mathematical Association of America college algebra standards (Mathematical Association of America, 2011), the National Council of Teachers of Mathematics Standards (National Council of Teachers of Mathematics (NCTM), 2000), and the American Mathematical Association of Two-Year Colleges Beyond Crossroads standards (Wood, Bragg, Mahler, & Blair, 2006). However, while these standards stress the importance of conceptual understanding, their detailed explication of what students should learn tends to focus on computational tasks (e.g. being able to perform function composition) rather than on the specific conceptual ways of thinking that underpin those tasks (e.g. having a process view of function).

Teacher beliefs and expertise

The relationship between teacher beliefs and practice is complex; for example, teachers do not always employ teaching practices that strongly reflect their professed beliefs about how students learn. However, despite this complexity, there is significant research suggesting that teacher beliefs are often strongly related to the teaching practices that teachers implement in the classroom, and therefore are also related to student beliefs and learning experiences (see e.g. (Fang, 1996; Maggioni & Parkinson, 2008). So understanding teacher beliefs is one critical component of understanding instructor practice and its impact on student learning.

On the other hand, teacher expertise also has the potential to benefit the research community by contributing important information about what teachers have learned while teaching; this knowledge can then be used by researchers to generate and test new theories about how students learn and about what is effective in the classroom. As Schulman (1987) explains, “One of the more important tasks for the research community is to work with practitioners to develop codified representations of the practical pedagogical wisdom of able teachers” (p. 11).

This study uses a *teacher-as-researcher* interpretation of *action research*, as originally coined by Stenhouse (1975) and later expanded conceptually by Elliot (1991) and then Cochran-Smith and Lytle (1993; 2009). In this framework, teacher-practitioners investigate research questions not only to improve their own practice, but also to add to a larger body of knowledge than can be implemented by other teachers in similar contexts. This is a more inclusive view that includes practitioner experiences as a valid foundation for knowledge production.

Theoretical Framework

This study uses Vygotsky’s (1986) theory of concept formation as a framework for investigating student understandings in elementary algebra. According to Vygotsky, algebraic symbols, graphs and other representations of mathematical objects and concepts mediate two interconnected processes: 1) the development of a mathematical concept in the individual; and 2) the individual’s interaction with an external mathematical world where these representations are rigorously codified. Learners begin to use these representations before they have “full” understanding of their meaning, and it is through this experimentation and attempts at communication with “more knowledgeable” others over time that they internalize more formal and correct meanings for the objects that the representations symbolize.

Methodology

Five elementary algebra instructors collaborated on this action research project, some of whom are also active educational researchers. The group included faculty with doctorates in both mathematics and mathematics education, who have taught at the high school, community college, and university levels. Faculty came from varied backgrounds, and included both men and women, several different racial/ethnic, national, and immigrant backgrounds, and reflected a variety of different teaching styles.

Conceptual Framework

This study used the *Action Research Spiral* Framework (Kemmis & Wilkinson, 1998) to guide the process of collaborative exploration into student thinking about elementary algebra concepts. This framework outlines a cyclical practice in which practitioners cycle through the following steps repeatedly in a spiral: 1) plan; 2) act and observe; 3) reflect; 4) revised plan, etc.

First each instructor independently created a list of concepts that they saw as fundamental to elementary algebra. After all instructors had created their own list, all lists were combined. Then a series of discussions ensued, during which various topics and sub-topics on the original master list were combined, rephrased, removed, added, and otherwise revised. Instructors used the framework to inform the creation of assessments and classroom activities, used these in their classes, and then used their experiences to inform revisions in a cyclical process over four separate semesters.

In deciding on what concepts to explore, instructors were asked to think not just about the current elementary algebra course that they were teaching, but about tertiary elementary algebra in general, including variations in what might be included on the syllabus at different colleges. The syllabi of elementary algebra courses at a number of different colleges in the U.S. were consulted to give instructors an idea of the range. Several framing questions were used both during initial independent selection of topics and the subsequent discussions:

- What concepts would we want students to still understand a year after they have completed an elementary algebra course?
- What fundamental algebra ideas are necessary for future mathematics courses (e.g. college algebra, pre-calculus, calculus)?
- What fundamental ideas are of significant value in other liberal arts math courses (e.g. statistics), or necessary in order to be able to apply algebraic thinking in “real” life (e.g. financial calculations, risk calculations)?
- For which particular concepts might you conclude that students had missed the “whole point” of elementary algebra if they were to finish the course without understanding them?

Results

In the process of identifying a conceptual framework for elementary algebra, the group first identified a list of four broad types of tasks that they felt all successful elementary algebra students should be able to complete, whatever the differences across elementary algebra curricula (see Table 1). Then, using these tasks as an initial frame, the group developed a list of concepts that, based on their cyclical experience interacting with students in elementary algebra classes, they felt to be fundamental in order for students to successfully complete these tasks (see Table 2 for a partial reporting of that framework).

Table 1. Fundamental Elementary Algebra Tasks
1. Expressing and correctly interpreting relationships, patterns, and properties in expression or equation form, through correct use of algebraic symbols
2. Simplifying expressions by replacing them with equivalent expressions
3. Solving an equation/inequality or system of equations and correctly interpreting the solution set
4. Relating an equation, or properties of an equation, to a graph, and vice versa
This list of tasks was not intended by the instructors to include all of those that might be relevant to any elementary algebra class given in any context.

The instructors theorized that the fundamental elementary algebra tasks in Table 1 could only be completed if students understand the following concepts in the Framework given in Table 2.

<table>
<tr><th>Table 2. Elementary Algebra Concept Framework (selected topics presented in detail)</th></tr>
<tr><td>1. Algebraic Symbolism: Understands how to express relationships and patterns in expression or equation form, and can explain in words the pattern or relationship expressed by a given expression or equation
(The sub-concepts in this concept are not outlined here because of space constraints.)</td></tr>
<tr><td>2. Algebraic Structure: Recognizes algebraic structure, with respect to the relevant context of a particular problem-solving goal
a. Understands the role of a variable:
i. That it can take on a wide variety of values;
ii. That its value can vary or that it can represented a fixed unknown;
iii. That any expression can be substituted in for a variable;
iv. That a variable functions as a set of parentheses around whatever is substituted into its place;
v. That every instance of a variable stands in for the same value;
vi. That any repeating expression can be replaced by a variable, as long as that variable is defined to take on the original value that it replaced.
vii. That during substitution, the structure of the expression outside the part being replaced remains independent of and unchanged by whatever is being substituted into it
b. Is able to view expressions or equations with respect to a particular context, and in this context identify the relevant properties
(The sub-concepts for 2.b. are not outlined here because of space constraints.)</td></tr>
<tr><td>3. Properties/Generalizing Arithmetic Operations: Understands the definitions of basic arithmetic operations and can use those definitions to describe general patterns and properties (in words and using equations).
For example, understands basic definitions of addition, subtraction, multiplication, division/fractions, positive whole exponents and square roots and can use these definitions to determine when these operations (or combinations of these operations) have certain properties.
(The sub-concepts in this concept are not outlined here because of space constraints.)</td></tr>
<tr><td>4. Equality/Equivalence: Understands equality/equivalence
a. Understands what it means for two expressions to be equal
i. Understands that two expressions are equal if and only if they are equal for all possible (combination of) values of (each of) the variable(s)
ii. Understands that if two expressions are equal, one expression may be substituted for the other in any other expression or equation
1. Understands that simplifying (or otherwise rewriting) expressions is a process by which one expression is replaced by another equivalent expression
b. Understands what it means for two equations to be equivalent
(The sub-concepts in 4.b. are not outlined here because of space constraints.)</td></tr>
<tr><td>5. Equations as Relationships between Variables: Understands equations with two variables as something that describes the relationship between two variables, describing how one variable varies with respect to changes in the other variable.
(The sub-concepts in this concept are not outlined here because of space constraints.)</td></tr>
<tr><td>6. Thinking Graphically: Understands how one and two-dimensional graphs describe the relationship depicted in</td></tr>
</table>

a particular equation or inequality and vice versa, and can describe how different operations will impact a graph. *(The sub-concepts in this concept are not outlined here because of space constraints.)*

In a cyclical process of experimentation, instructors developed assignments and assessment questions intended to either 1) assess the extent to which students understood one or more concepts listed on the framework; or 2) confront students with tasks that would require them to directly engage with a common misconception or a type of productive struggle that might better reveal their current understandings related to one of the concepts on the framework, or how they use those understandings to complete specific elementary algebra tasks.

An example of one question in the first category is below:

Assume that $a \neq 0$. Dale simplifies the expression a^3a^{-2} and gets the correct expression a. Which of the following must be true? There may be **more than one** correct answer—select **ALL** that are true.

a. $a^3a^{-2} = a$
b. If Dale lets $a = 10$ in both the expressions a^3a^{-2} and a, he will get two different answers.
c. Dale can substitute a for a^3a^{-2} anywhere it appears.
d. If Dale lets $a = 20$ in both expressions, he will get the same value for each expression.
e. Dale needs to know the value of a before he can say whether a^3a^{-2} and a are equal.

This question was designed to test the extent to which students understand the items listed under 4.a. in the framework in Figure 2. Based on the answers that students gave, instructors could then engage with students about their understanding of specific components of item 4.a. in order to better understand what those are and how they relate to one another. Based on conversations with students as a result of this question, the framework was revised: The first framework draft contained only item 4.a.ii.1; after repeated cycles of the research process the additional items under 4.a. were added and structured hierarchically.

An example of a task that falls into the second category is the following question, which one of the instructors used for an in-class activity:

Suppose there is a new algebraic operation called the bow tie, defined this way: $\bowtie a = \frac{1}{a} - a^2$
Use this definition to rewrite the following expressions (no need to simplify afterwards!) so that they no longer contain a bow tie symbol:

a) $\bowtie (-2)$
b) $\bowtie (x^2)$
c) $\bowtie \left(\frac{1}{y}\right)$

Instructors expected students to make the following mistakes somewhat frequently: $\bowtie (-2) = \frac{1}{-2} - 2^2$, $\bowtie (x^2) = \frac{1}{x^2} - x^2$, $\bowtie \left(\frac{1}{y}\right) = \frac{1}{y} - \left(\frac{1}{y}\right)^2$. The expectation was that students who made these mistakes did so through oversight—for example, they might forget to write both negative signs when substituting in -2 for $-a^2$. However, in one-on-one discussions with students, it became clear that many students did not forget to write the negative sign in this case; rather, they believed that because the -2 already had a negative sign, that this negative sign took the place of the one that was already in the expression $\frac{1}{a} - a^2$. This led to a revision of Framework item 2.a. to include sub-item vii.

Discussion and Plans for Next Steps

This framework reflects only the experience of one group of elementary algebra instructors at the college level and may not reflect the experiences of all tertiary elementary algebra instructors. This is also just one step of data collection in our larger goal of developing a conceptual framework for elementary algebra at the college level. Another ongoing study that is a part of this larger project is an extensive literature review synthesis whose goal is to develop a comprehensive research-based conceptual framework for elementary algebra that is based on existing literature from primary and secondary settings. The next step will then be to compare that framework to the instructor-generated framework in order to identify areas where they overlap and where they do not. Areas where the frameworks differ will then be used as a starting point for future research projects investigating how the algebraic understandings and learning processes of tertiary students in elementary algebra may differ from the experiences of primary and secondary students learning similar content.

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