An Investigation of the Development of Partitive Meanings for Division with Fractions: What Does It Mean to Split Something into 9/4 Groups?

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In this paper we describe a study involving twelve preservice elementary teachers who were attending a community college. The design and implementation of this study were guided by the research question: In what ways do students reason through a sequence of tasks which progressively become more abstract, and which challenge primitive intuitions regarding partitive division? We highlight students’ ways of thinking involved with division that are not easily generalizable, that favor numerical procedures over quantitative reasoning, and which are obstacles to the development of more robust meanings for division.

Keywords: Fraction Division; Partitive Division; Preservice Elementary Teachers

Discussion of the Literature

Researchers (Fischbein, Deri, Nello, & Marino, 1985; Harel, Behr, Lesh, & Post, 1994; Rizvi & Lawson, 2007; Simon, 1993; Thompson & Saldanha, 2003; Tirosh, 2000) have long acknowledged some primitive ways of thinking about partitive and quotitive division. Partitive division is when \( a \div b \) is interpreted as the amount of the quantity referenced by \( a \) per one unit of the quantity referenced by \( b \), given that the quantities are proportionally related. One primitive model for partitive division identified by researchers is fair sharing, which is characterized by thinking about splitting \( a \) into \( b \) equal parts. This primitive model requires the divisor to be a whole number, and thus the value of the quotient should be less than the value of the dividend; in other words, division makes smaller. Quotitive division is when \( a \div b \) is interpreted as the number of copies of \( b \) that make \( a \), which can also be interpreted as the measurement of \( a \) in units of \( b \). Concerning quotitive division, the primitive model of repeated subtraction requires the divisor to be smaller than the dividend. These primitive models for division are rooted in reasoning with whole numbers and they continue to influence the reasoning of students and teachers, even after they are exposed to more sophisticated models (Fischbein et al., 1985). In particular, these primitive models obstruct sensible reasoning pertaining to division involving fractional values.

In a study of preservice elementary teachers, Simon (1993) noticed that the subjects could accurately execute procedures for long division of whole numbers, but that these procedures were not well connected to the subjects’ meanings for division. He stated that “their lack of conceptual understanding given their algorithmic competence seems to challenge the idea that procedural practice eventually leads to understanding (Simon, 1993, p.249).” In a study of preservice elementary teachers, Tirosh (2000) investigated the impact of primitive models on division involving fractional values. She observed that the subjects heavily relied on procedures, such as flip and multiply, instead of relying on meanings for division. She introduced the subjects to formal justifications for the flip and multiply algorithm, but these formal arguments were based largely on symbolic manipulation of non-contextualized variables, and it was unclear whether these justifications would be accessible to elementary students. Rizvi and Lawson (2007) noticed that none of the 17 preservice teachers from their study could explain the flip and multiply algorithm, nor could any of the subjects initially pose a word problem that required division by a fractional value. The researchers attributed these deficiencies to a reliance on the primitive fair sharing and repeated subtraction models for division.
We have found little research on partitive meanings for division when the divisor is a fractional value. A quick survey of textbooks and online resources is likely to reveal that many attempts to connect numerical division of fractional values to a meaning for division are based on the quotitive division model. However, some researchers and educators (Beckmann, 2011; Gregg & Underwood Gregg, 2007; Kribs-Zaleta, 2008; Ott, Snook, & Gibson, 1991) have illustrated partitive models for division with fractions, but they do not discuss the development of these meanings. This paper reports on a study designed to investigate the development of partitive meanings for division with fractions.

The Partitive Model: What Does it Mean to Split Something into $\frac{9}{4}$ Groups?

The design of this study assumed the subjects had existing meanings for fair sharing with whole numbers of groups, as well as meanings for fractions as operators. As such, we intended to build on these meanings by introducing the subjects to situations that require partitive division with whole divisors, followed by situations with non-whole divisors. Let’s consider a learning progression that begins with primitive meanings for partitive division with whole divisors, and ends with robust meanings that accommodate non-whole divisors. Consider 6 cups of water fitting perfectly in 3 equally-sized whole containers. The relative size of one container’s capacity to the total amount of water is critical – if 6 cups fit into 3 equally-sized containers, then one container holds $\frac{1}{3}$ of the 6 cups. This way of thinking forms a meaningful foundation for the numerical equivalence of the expressions $6 ÷ 3$ and $1/3 \times 6$. Next, suppose 6 cups of water fit perfectly in $\frac{9}{4}$ containers. Interpreting $\frac{9}{4}$ containers as 9 quarter-containers allows one to reason that $\frac{1}{9}$ of 6 cups will be in each quarter container with four copies constituting the capacity of one whole container (see Figure 1). Thus a whole container’s capacity would be $4 \times \frac{1}{9} \times 6$, or $4/9 \times 6$ cups. This idea yields a numerical equivalence between $6 ÷ \frac{9}{4}$ and $\frac{4}{9} \times 6$. As another example, consider a situation where 6 cups of water fit perfectly in $\frac{2}{3}$ of a container. Thus, $\frac{1}{3}$ of the container holds $\frac{1}{2}$ of 6 cups, and the whole container holds 3 times as much as $\frac{1}{2}$ of 6 cups (see Figure 2). This way of thinking yields a numerical equivalence between $6 ÷ \frac{2}{3}$ and $\frac{3}{2} \times 6$. The study described in this paper investigates the development of these schemes for partitive division involving fractions.

![Figure 1. 6 cups in 9/4 containers.](image1)

![Figure 2. 6 cups in 2/3 of a container.](image2)

Methodology

This study focused on twelve elementary education students in a Mathematics for Elementary Teachers course at a community college. An instructional unit, focused on the meanings of
division, was implemented in the course and spanned three class sessions. Of the five learning objectives for this unit, two are the focus of this paper: (1) Use partitive and quotitive meanings for division (instead of algorithms) to divide by rational numbers and (2) Make sense of the *flip and multiply* procedure by using the partitive meaning for division. We issued a total of four assignments to each of the twelve students in the class. The first assignment preceded any formal class discussion on division and the remaining three assignments were given out after each class session. Two students, one higher-performing and one lower-performing, were selected to participate in videotaped interviews while they worked through the assignments. Leveraging Goldin’s (2000) principles, the interviews with these two students were semi-structured and task-based, with the purpose of investigating their thinking. We asked the other ten students to each work on their assignments alone and without resources, to do the tasks from each assignment in order, and to clearly present their solutions in writing. The design of this investigation and subsequent data analysis were guided by the research question: *In what ways do students reason through a sequence of tasks which progressively become more abstract, and which challenge primitive intuitions regarding partitive division?*

**Discussion of the Data**

During this study, the subjects participated in a variety of tasks. In this paper, we narrow our discussion to the data from the following three tasks:

**Task 1:** Divide 27 gallons of water into 9/4 containers. How much water is in one whole container?

**Task 2:** Suppose an unknown amount of water is divided into 9/4 containers. What could you say about how much water is in one whole container?

**Task 3:** Explain why it is that when you divide by a fraction, you can multiply by the reciprocal of the fraction instead. In other words, explain the following: \( a \div \frac{b}{c} = a \cdot \frac{c}{b} \)

We designed these tasks to be successively more abstract. For Task 1, we considered a response to be correct if it was *12 gallons of water in one whole container*. For Task 2, we considered a response to be correct if it was conceptually equivalent to saying *4/9 of the water is in one whole container*. For Task 3, we considered a response to be correct if it was a generalization of valid thinking from Tasks 1 and 2, or some other valid explanation. Table 1 summarizes our analysis of the subjects’ responses to these three tasks, and it reveals that as the tasks become more abstract, the students became less successful overall. We now discuss the thinking of the two students who were videotaped. We will refer to them as Adam and Sue.

<table>
<thead>
<tr>
<th></th>
<th>Task 1</th>
<th>Task 2</th>
<th>Task 3</th>
</tr>
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<tbody>
<tr>
<td>Number of students with a correct response</td>
<td>8</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Number of students with an incorrect response</td>
<td>3</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Number who said “I don’t know” or gave no response</td>
<td>1</td>
<td>3</td>
<td>7</td>
</tr>
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**The Case of Adam**

Adam was the only student out of the twelve subjects who demonstrated valid reasoning during all three tasks. For Task 1, he drew 9/4 containers, then decided to procedurally compute \(27 \div 9/4\) by inverting and multiplying to get 108/9, which he reduced to 12 through procedural division. He admitted to using the numerical operation of division because the word *divide* was
in the prompt. Once he calculated that the answer should be 12, he decided to find the amount in each quarter-container, by dividing 12 by 4 to get 3. He did not indicate that he could have also divided 27 by 9 to get 3. In fact, his calculation of $27 \times 4/9$ by first multiplying 27 by 4 is indicative that he was operating numerically and not thinking quantitatively about partitioning and iterating. He then confirmed that a total of 27 gallons was in the 9/4 containers by saying that 12 gallons were in each of the two whole containers and 3 more gallons were in the remaining quarter-container (see Figure 3).

![Figure 3. Adam’s response to Task 1.](image)

During the interview, Adam’s language was not consistent. He sometimes referred to each quarter-container as a whole container, as well as misspeaking about other referents. However, we suspect that his language was simply misrepresenting his valid thinking. Figure 3 also reveals that Adam labeled one container as “4/9”, and he said that “4 out of the 9” pieces make one whole container. As such, and not surprisingly, for Task 2 he wrote that “4/9 of the whole” corresponded to one whole container. For Task 3, Adam demonstrated that he was beginning to generalize his thinking. He attempted to explain the algorithm by describing division by 7/2, as depicted in Figure 4. When pressed to speak in terms of $a$, $b$, and $c$, Adam responded by describing partitive thinking, but then gave a quotitive example. He said “The whole thing is the $a$...and I’m cutting it up to a certain amount of pieces. I don’t know how many… abstract… but let’s say it’s one third. And I want to know how many one thirds fit into it [referring to the whole].” Ultimately, Adam was unable to generalize the partitive division models that he had earlier demonstrated using specific fractional values for the divisor.

![Figure 4. Adam uses division by 7/2 to illustrate the invert and multiply procedure.](image)

**The Case of Sue**

Sue appeared to have no trouble with Task 1. She drew nine contiguous boxes, marked “3” on each, indicated that four such boxes made a whole container, and concluded that 12 gallons were in one whole container. This is depicted in Figure 5.
On Task 2, Sue was immediately perturbed. She said she didn’t know what to do, pointing out that the amount of water was not provided like it was in Task 1. The researcher asked what she would do if the amount of water was 18 cups. She proceeded to answer this question in the same way that she did for Task 1. The researcher then asked her to consider what she would do if “x cups” were the total amount of water, but she was ultimately unable to respond. Since Sue was stumped, the researcher returned to Task 1 and asked her how she knew to put a “3” in each box. She explained that she knew the total had to be 27 gallons and that there were nine boxes. She mentioned that each of the nine boxes had to have the same amount of water and mentioned a guess and check strategy. Two gallons per box was too little (“nine times two is only 18”), but nine times three gives the correct 27 gallons. The researcher then asked her why she answered that 12 gallons were in one whole container. She said that each box was one fourth of a container, so she added the four copies of three gallons to get 12 gallons. The researcher then drew nine contiguous boxes, shaded in four of them, and asked Sue how much of the entire collection was shaded. Sue promptly answered “four ninths”. The fact that Sue answered “four ninths” so quickly in the latter situation indicates that Sue’s meanings for fractions are limited to the out of model and that she does not have developed meanings for fractions as operators. As such, her schemes for solving Task 1 were not generalizable to the point where she could sensibly talk about Task 2. For Task 3, Sue did not know how to respond.

Conclusion

Primitive ways of thinking about division continue to be pervasive in mathematical thinking. This research explored the development of more robust meanings for partitive division, which are not thwarted by non-whole divisors. The initial data reveals that underdeveloped meanings for fractions are impediments to the maturation of robust meanings for division. For example, Sue mentioned “four ninths” when she saw 4 out of 9 boxes shaded, but she did not appear to think of four ninths as an operator on the total amount of water. The data also suggests that quantitative operations, such as partitioning and iterating, are often neglected in favor of procedural approaches to division. This was illustrated by Adam, in Task 1, when he procedurally calculated that 12 gallons were in each whole container; yet, there was no indication that he partitioned the total amount of water into 9 equal pieces. Additionally, we see examples of schemes for division that are not generalizable to more abstract levels of meaning. For example, Sue’s guess and check scheme in Task 1 was dependent on knowing the total amount of water. It is evident that the development of partitive meanings for division with fractions depends on more robust meanings for both fractions and division. Additional research is needed to better investigate this claim.

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References


