

## Beyond the Product Structure for Definite Integrals

Courtney Simmons  
Oklahoma State University

Michael Oehrtman  
Oklahoma State University

*Over the past decade research has shown that a Riemann sum based interpretation of the definite integral supports a robust understanding of the underlying structure of the integrand/differential relationship and facilitates students' ability to make sense of contextual integral models. However, current studies center this understanding on the multiplicative structure  $f(x) \cdot \Delta x$  which does not account for many practical uses of integration. In many situations, the  $\Delta x$  is most productively conceived as a component of another quantity which might then be incorporated in any of a variety of quantitative models, such as an inverse square law rather than a simple product. To fill this gap, this study utilized Dewey's theory of inquiry to identify three interpretations of the definite integral which proved productive for students when modeling definite integrals that extend beyond the traditionally studied product structure.*

**Key words:** Definite Integral, Riemann Sum, Adding Up Pieces, Quantitative Reasoning, Physics and Engineering

### Introduction and Literature Review

In 2001 Sherin lamented “it seems reasonable to suggest that, prior to physics instruction, students should learn to invent at least some simple types of mathematical models and to express the content of those models.” This sentiment is indicative of the general shift in attitude towards the calculus curriculum in recent years proposed by education researchers; that calculus curriculum should focus on an introduction to the concepts of limit, differentiation, and integration in a way that supports students' acclimation of mathematical ideas into their applied STEM fields, simultaneously leveraging that contextualization to support students' interpretation, mathematization, and argumentation (Jones, 2013, 2014, 2015b; Meredith & Marrongelle, 2008; Oehrtman, 2009; Redish & Smith, 2008; Sealey, 2006, 2014; Sealey & Oehrtman, 2005; Thompson, 1994; Thompson & Silverman, 2008).

Although differentiation and limits are both important aspects of the calculus curriculum, special attention should be paid to the definite integral for both its utility within mathematics, as well as its application to modeling in real world contexts (Hibbeler, 2004; Sealey & Oehrtman, 2005; Thompson & Silverman, 2008). Sealey (2014) provided a framework for characterizing student understanding of Riemann sums and definite integrals, decomposing this understanding into five parts; orientation to the constituent quantities, a product layer in which one attends to the quantitative relationship between  $f(x_i)$  and  $\Delta x$ , the summation of these products, a limiting process applied to this summation, and a function layer in which ongoing accumulation may be considered. Von Korff and Rebello (2012) further adapted this framework into a physics context promoting the concept of infinitesimal products. Jones (2013) applied Sherin's (2001) symbolic forms to students' interpretations of the definite integral, classifying their reasoning into categories such as area under a curve, function matching, and adding up pieces. Jones (2015) later refined his description of the adding up pieces symbolic form and described a multiplicatively-based summation (MBS) conception that was “highly productive” in engaging students in both the mathematical structure of the definite integral as well as in modeling for

physics-based contexts. This Riemann sum interpretation of the definite integral  $\int_a^b f(x) dx$  focuses on adding up many terms derived from a multiplicative relationship between a (possibly) varying integrand  $f(x)$  and small or infinitesimal differential  $dx$ . The products in this conception must be abstracted from a parallel multiplicative relationship in what we call a “basic model” between a constant quantity  $F$  and a (possibly) larger value  $X$ , respectively. Such decompositions to the Riemann sum structure have been stressed by math, engineering, and physics education researchers as essential for STEM student success, including mathematics majors (Doughty, McLoughlin, & van Kampen, 2014; Jones, 2013, 2014; Meredith & Marrongelle, 2008; Redish & Smith, 2008; Sealey, 2014; Sherin, 2001).

Meredith and Marrongelle (2008) made the apt point that a product structure derived from a basic model  $F \cdot X$  to generate terms of the form  $f(x) \cdot \Delta x$  in a Riemann sum is only adequate for modeling accumulation of a quantity when the integrand is a rate of change or density, yet many contexts do not naturally decompose into this quantitative structure (pp. 575-576). Oehrtman (2015) showed that an MBS interpretation was insufficient for students constructing more complex definite integrals from basic models that are not a simple product. To clarify reasons for students’ success, Oehrtman also analyzed the symbolic forms students applied to the basic model, how they transferred that quantitative reasoning to terms of a Riemann sum or the differential form of a definite integral, and how those symbolic forms interacted with the students’ symbolic forms for the integrals. Thus, only investigating the MBS conception of the definite integral leaves a potentially significant gap in our understanding of the quantitative reasoning required for successful modeling with definite integrals and interpretations of differential forms apart from a generalized product of integrand and differential. Our study seeks to further explore the ways in which students conceptualize definite integrals that do not match the traditional  $f(x) \cdot dx$  multiplicative structure and identify key aspects of their reasoning which lead to productive results. Specifically, we pose the research question:

What interpretations of a definite integral and basic model are productive and unproductive for students as they progress through increasingly complex definite integral modeling tasks?

### **Theoretical Perspective & Methods**

As this study seeks to identify how students engage in and overcome difficulties modeling definite integrals beyond the product structure they typically first learn, we chose to employ John Dewey’s theory of inquiry (Dewey, 1938; Hickman, 1990) to their problem-solving efforts. Dewey characterized knowledge as a product of the activities a student actively engages in when encountering non-routine problems. These activities involve strategically identifying mental tools with which to make sense of the problem, testing that tool in the context of the problem, and utilizing the tool to clarify the problematic situation in some way for the student. New knowledge emerges through the dialectic interplay between application of the tool against the problem and evaluation and refinement of the tool against the perceived progress. With this interpretation of learning in mind, we will classify an interpretation, i.e. tool, of the definite integral as productive if it is perceived by the student in advancing their progress towards resolving a problematic situation. We note that a productive interpretation need only clarify the situation for the student and does not have to be consistent with the researchers’ own views for resolving the problem.

Four pairs of students were selected to participate in an hour and a half long interview focusing on physics applications of the definite integral. The choice to work with pairs rather than individual interviews was made so students could discuss their ideas with each other, rather than an authority figure. The students were encouraged to discuss their thinking aloud while working through problems on a whiteboard until they were satisfied with their solution, after which the interviewers would ask follow-up questions to clarify any notes made during the encounter.

Because our study seeks to understand students' progress beyond an integrand×differential quantitative relationship, it was important to recruit students who were likely already familiar with the MBS conception described by Jones. As such, Group C consisted of a senior undergraduate and second-year masters student who had recently taken an Advanced Calculus course, while the remaining three groups, A, B, and D, were recruited from a second semester calculus course in which the instructor utilized a calculus curriculum emphasizing the Riemann sum approach for definite integrals in classroom labs (Oehrtman, 2016). After our initial round of interviews, Group B was asked back for a second one hour interview due to the strength of their collaboration and quality of data provided. These interviews were video-taped and later transcribed for coding.

Each group was first asked to discuss their general interpretation of a definite integral, followed by a series of tasks chosen to reveal their reasoning while modeling definite integrals as they progressed from simple rate×time contexts to situations that obscure the product structure of a differential form (Figure 1). The intent was to track the students' operative interpretations of the definite integral, the components that comprise the integral, the quantitative relationships involved in the basic physics formulas, and the students' perception of the interconnection between these concepts. We anticipated many students would approach some of the simpler problems heuristically, avoiding reasoning quantitatively, but that the progression of tasks would require an eventual shift to adding up pieces in progressively sophisticated ways in order to be successful.

1. When fully wound, a toy car will travel in a straight line for just over 17 seconds. Its speed  $t$  seconds after it is released is  $v(t) = 3 \sin\left(\frac{t^2}{100}\right) m/s$ . Write an integral that gives the distance the fully wound car will travel during the first 10 seconds it travels.
2. Linear density is a measure of an objects mass per unit length. This means for example if an object has length 3.2 ft and **constant** linear density of . 5 lb/ft it will have a total mass of  $(3.2)(.5) = 1.6$  lbs.  
King Arthur's lance is 10 feet long and heavier at one end, decreasing from 1.2 lb/ft at the base to 0.2 lb/ft at the tip. That is, the linear density  $x$  feet from the base is  $\delta(x) = 1.2 - 0.1x$ . Write an integral that gives the mass of King Arthur's lance.
3. When pollen from a red cedar tree is released from their cones it travels through the air. Pollen from a mature tree settles on the ground with an estimated density of  $\delta(r) = \frac{37}{10+r} g/m^2$  a distance  $r$  meters from the tree.  
Write an integral that gives the mass of pollen distributed within 100 meters of a mature tree.
4. Coulomb's law says that the force between two electrically charged particles is  $F = k_e \frac{q_1 q_2}{r^2}$ , where  $k_e$  is Coulomb's constant ( $k_e = 8.99 \times 10^{21} N \cdot m^2 / (\mu C)^2$ ),  $q_1$  and  $q_2$  are the charges of the particles in microcoulombs ( $\mu C$ ), and  $r$  is the distance between the charges. Coulomb's law only directly applies to two point charges.  
Write an integral that gives the electrostatic force between a uniformly distributed charge of 29 microcoulombs on a rod of length 7 meters and a particle with charge of 2 microcoulombs 3 meters from the rod along its axis.
5. The energy measured in joules to lift an object straight up  $h$  meters is  $E = M \cdot g \cdot h$ , where  $M$  is the mass of the object measured in  $kg$ , and  $g$  is the gravitational force measured in  $m/s^2$ .  
Suppose a 10-meter chain with a total uniform mass of 15kg is freely hanging from the roof of a building. Write an integral that represents the total energy required to lift the chain to the top of the building.

Figure 1. Task Sequence.

Group A was only able to progress through task 2, while the remaining three groups were able to advance to the fourth task with varying levels of success. Task 5 was chosen as a supplemental question for Group B to work through during their second interview session and was not presented to any other group.

The video transcriptions were coded using open and axial coding methods of grounded theory (Strauss & Corbin, 1998). Because we were interpreting this data through the lens of Dewey's theory of inquiry, the initial coding pass sought to identify problematic situations for students, in the sense of non-routine engagement and reciprocal influences of their evolving conceptions for definite integrals and basic models with their understanding of the modeling tasks. After these instances were identified, each was broadly coded for all cognitive tools utilized, obvious relationships to previous research, and any specific tools which lead to productive results. Subsequent classifications looked to identify associations between these codes both throughout a specific interview, as well common themes amongst the different groups to develop a narrative regarding students' conceptualizations of these definite integrals. Common problematic situations and tools which led to productive results were identified and the information was synthesized into 3 defining characteristics of productive modeling of the definite integral, Adding up Pieces, Quantitative Reasoning, and Utilization of "nearly constant," which will be discussed in the following section.

## Results

### Adding up Pieces

Echoing previous research, our findings suggest that conceptualizing a definite integral as a summation allows for students to model physics equations in a way consistent with both mathematical and physics/engineering constructs. However, our interpretation of adding up pieces differs slightly from that of previous research as we allow for integration of quantitative relationships more complex than the traditional  $f(x) \cdot dx$  structure. Due to the nature in which we recruited our subjects, all pairs demonstrated at least some underlying Riemann sum structure for the definite integral. Group A showed the weakest correlation between Riemann sums and definite integrals, often relying on area under the curve symbolic forms or graphical interpretations, although when pressed they could describe these interpretations in the context of representative rectangles (Jones, 2013, 2015a). This weak connection to adding up pieces played a role in Group A only completing two of the tasks, and only the first completely correctly.

### Quantitative Reasoning

Aside from the adding up pieces conceptualization, quantitative reasoning is perhaps the most crucial aspect of modeling definite integrals (Thompson, 1994, 2011; Thompson & Silverman, 2008). Within the construct of a definite integral which is comprised of a non-routine quantity, there are multiple layers of quantitative reasoning a student must navigate. First, the student must attend to the quantitative relationship of the definite integral itself. Within that structure the student must also attend to the relationship between the integrand and differential. Certain contexts lend themselves to obvious relationships, such as Tasks 1 and 2, however, the more complicated structures have yet another quantitative relationship layer; how the differential is situated within a quantitative formula. In Task 3 this extra consideration did not prove overly difficult for most groups, however, in Task 4 it caused numerous problematic encounters for the students and required significant effort to resolve. We note that there are often multiple ways to reason about the basic model within a definite integral quantitatively. Figure 2 displays three

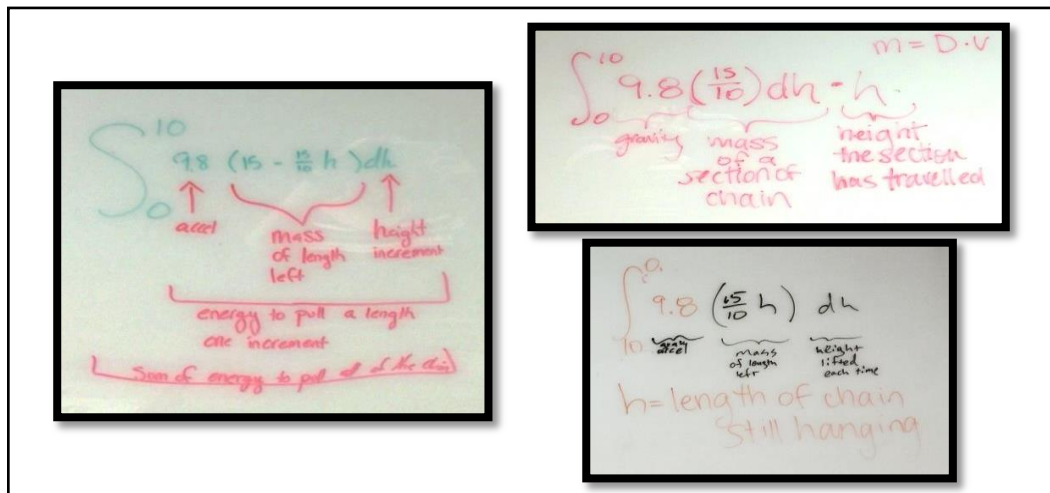


Figure 2. Three quantitative relationships demonstrated by group B in Task 5.

different ways in which group B conceptualized Task 5 during their second interview; the top two were simultaneously modeled by each student individually while discussing the problem relatively fluently with one another. In light of this encounter, we conjecture that students need not conceptualize the quantitative relationship within a basic model in a specific way to be productive in modeling a definite integral. Rather the importance lies in attending to the quantities that comprise a given formula and identifying their relationship to each other in a meaningful way, while simultaneously considering any underlying implications of the basic model, such as the multiplicative structure distance = speed  $\times$  time is only valid when speed is constant.

### Utilization of “Nearly Constant”

In each of the problems presented there is an underlying basic model which holds for constant values. Every group used these formulas as a part of their final integral structures, asserting that the reason you must integrate is that at least one of the quantities in question is varying. Every pair articulated some form of justification for why they used the formula for a “nearly constant” segment, as shown in Figure 3.

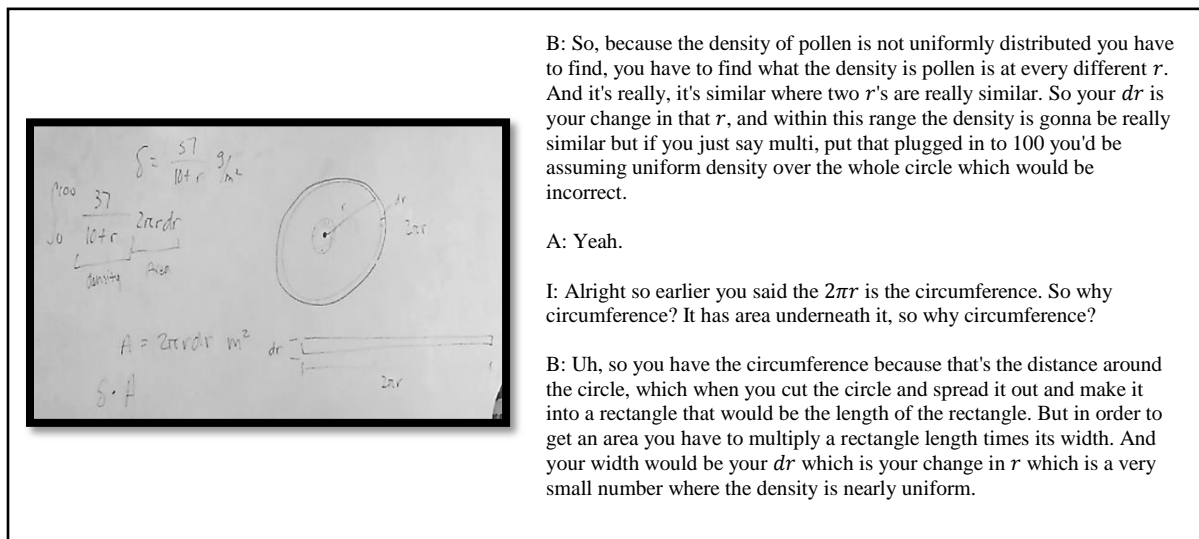


Figure 3. Group C discusses “nearly uniform” density in Task 3.

For many pairs, inserting the given formula into the integral during the first few tasks was a routine procedure that only became problematic and subsequently justified when questioned by the interviewers. This was especially evident in Task 1, where every group made reference to the relationship that velocity is the derivative of position. However, when pressed, most groups were able to identify that over a small duration of time, speed was basically constant, which allowed for a confident explanation of the role of the differential within the integral. This justification of process, which was evident in early tasks, was not as prevalent in the data for Task 4. Instead there was a shift of focus to “what’s changing,” which led to many unproductive encounters and in one case led to dividing by the differential (Figure 4).

Group B demonstrated a strong adding up pieces conception and quantitative reasoning skills in all previous tasks, and were fluidly speaking about breaking up the rod in Task 4 into segments and adding forces together to obtain a total force. So in all other aspects they were conceptualizing this definite integral in ways compatible with current research. However, the desire to find the moving part of the equation permeated their reasoning, hindering advancement. Although very problematic for the students, the resolution of this interpretation was only possible through an interviewer’s intervention. It is possible that an extra layer of abstraction provided additional difficulties for this pair, as their initial problem contained no specific quantities. Numerical quantities were provided halfway through their encounter with this problem and the task was updated to reflect specific values for future interviews.

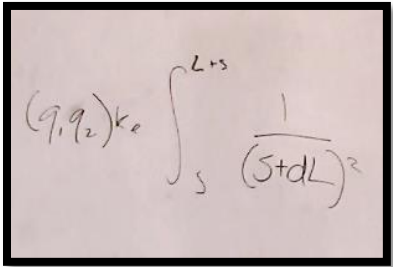
<p>J: And then we have 1 over <math>s^2</math>, that's what is changing, <math>s^2</math>?</p> <p>M: Yeah.</p> <p>J: So we need to change that to <math>(s + dl)^2</math>.</p> <p>M: Yes, cause <math>s</math> is the distance and the formula just has distance squared. I think that's it, but... that doesn't look like an integral to me.</p> <p>J: Yeah, <math>dl</math>'s down there and it's squared. That's not normal.</p>	
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Figure 4. Group B dividing by the differential in Task 4.

### Familiarity of context

Although not characterized as productive, and perhaps unsurprising, the unfamiliar context and quantities of electrostatic force proved problematic for the students in this study. Every group that attempted this task required at least some clarification of Coulomb’s law. The more troublesome complication arose when the groups tried to conceptualize the adding up process that would need to take place if they were to integrate. Unfamiliar with the principle of superposition, two of the groups envisioned the integral needing to account for forces compounding upon one another as the integrand ranged over its domain. An example of this is shown in Figure 5.

Because students’ lack of familiarity with superposition of forces was not intended to introduce complications, one of the interviewers stepped in to provide the supplementary explanation. This finding echoes Meredith and Marrongelle’s 2008 assertion that “understanding of the physical situation is necessary, but not sufficient for students to use their mathematical resources”.

J: From what I can understand, the function that they gave us,  $f$ , like this is just one member that we would get. This is the force between these two parts [points towards a diagram explaining Coulombs Law]. Over here we have a force between these two parts, and then a force between these two parts, and a force between these two parts, and we're adding all of those forces together.

M: Uhhh...

J: It's kind of like the other ones we were doing, it's like, the thing that's changing is the forces and that's what we're adding together.

M: Yeah...[laughs] but... I don't know. It doesn't make sense to me that it doesn't like compound, you know?

J: Compound?

M: Well cause wouldn't the charge of this be like, like right here have that plus all of this [points to the far end and then motions across the full length of the rod]. Wouldn't that have...

J: Ohhhh

M: That's why I was saying if it was moving it would be easier. Because then you're looking at this and then nothing but space. But you're looking at this [points to the third subsection] and then like that little part [motions to the two subsections preceding the third subsection].

J: I get what you're saying. So is it like a double integral?

M: [whispers] I don't know.

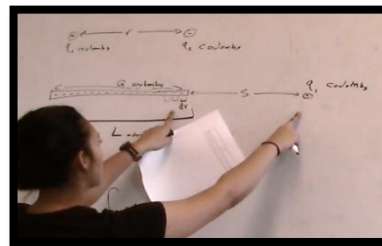


Figure 5. Group B discussing compounding force.

## Discussion

The results of this study illustrate three key interpretations of the definite integral, adding up pieces, quantitative reasoning, and attention to the quantity which is nearly constant, which we have identified as crucial for successful modeling of not only multiplicative basic models but also more general basic models found in many physics and engineering contexts. While discussed separately, these different themes are often interrelated and require the student to coordinate subtle interplay between them. For example, the decision to break a situation into multiple pieces to create a Riemann sum interacts with their conception of variable and constant quantities in the basic model and with their conceptions of the definite integral. As in previous research, we found the adding up pieces conception of the definite integral highly productive for students, however, when paired with the more complex basic model there are additional layers of this construct for the student to work through. It may be clear that the integral will add up small chunks of the basic model to obtain the desired total, while the quantitative reasoning behind breaking up the physical (or abstract) situation to obtain these small chunks can remain problematic for students. To overcome this difficulty, students must be able to attend to how the concept of a small chunk being “nearly constant” allows for the use of integration of the basic model, which can support their decision of how and why they are breaking up the situation into smaller constructs.

Based on the data collected in this paper, and the call for modeling of definite integrals in the other STEM fields, it is our contention that Jones’ (2015a) MBS conception of the definite integral can be productively extended to a Quantitatively Based Summation conception (QBS) for the definite integral, of which the MBS conception would be an often utilized subset. Similar to the MBS conception, this interpretation of the definite integral in a QBS is incorporated into a Riemann sum approach to the definite integral, but requires focus on the rich quantitative reasoning about the basic model and transferred in flexible ways to the differential form for the definite integral.

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