Examining Students' Procedural and Conceptual Understanding of Eigenvectors and Eigenvalues in the Context of Inquiry-Oriented Instruction

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This study examines students' procedural and conceptual understanding as evidenced by their written responses to two questions designed to assess aspects of their understanding of eigenvalues and eigenvectors. This analysis draws on data taken from 126 students whose instructors taught using a particular inquiry-oriented instructional approach and 129 comparable students whose instructors did not use this instructional approach. In this proposal, we offer examples of student responses that provide insight into their reasoning and summarize broad trends observed in our quantitative analysis. In general, students in both groups performed better on the procedural item than on the conceptual item. Additionally, the group of students who were taught with the inquiry-oriented approach outperformed the group of students who were taught using other approaches.

Key words: eigenvalues, eigenvectors, linear algebra, inquiry-oriented, student thinking

Linear algebra is a mandatory course for many science, technology, engineering, and mathematics (STEM) students. The theoretical nature of linear algebra makes it a difficult course for many students because it may be their first time to deal with this kind of abstract and conceptual content (Carlson, 1993). Carlson (1993) also posited that this difficulty arises from the prevalence of procedural and computational emphases in students' coursework prior to linear algebra, and that it might therefore be difficult for students to connect new linear algebra topics and their previous knowledge. To address this issue, researchers have developed inquiry-oriented instructional materials and strategies to help students develop more robust, conceptual ways of reasoning about core topics in introductory linear algebra (e.g. Wawro, Rasmussen, Zandieh, & Larson, 2013). In this proposal we examine assessment data to identify ways in which students reasoned about eigenvectors and eigenvalues. In particular, we identify differences in the performance of students whose instructors taught with a particular inquiry-oriented approach to teaching eigenvectors and eigenvalues and comparable students whose instructors did not use this approach.

In this work we draw on data from an assessment that was developed to align with four core introductory linear algebra concepts addressed in the IOLA instructional materials: linear independence and span; systems of linear equations; linear transformations; and eigenvalues and eigenvectors (Haider et al., 2016). The focus of this study is to identify the ways students understand and reason about eigenvalues and eigenvectors. In the assessment, there were two questions, question 8 and 9 that addressed eigenvalues and eigenvectors. Question 8 was a procedural item related to the eigenvalue of a given matrix and question 9 focused on the conceptual understanding of the eigenvectors. The research questions for this proposal are:

- How did students reason about eigenvectors and eigenvalues in the context of questions designed to assess aspects of student's procedural and conceptual understanding?
- How do the performance and ways of reasoning of students whose instructors adopted an inquiry-oriented approach to teaching linear algebra compare to the performance and ways of reasoning of other students?

Literature & Theoretical Framing

Many have argued that the shift from a predominantly procedural approach to mathematics many students experience before college to a more conceptual approach causes a lot of difficulties for students as they transition to university mathematics; linear algebra is a topic in which students struggle to develop a conceptual understanding (Carlson, 1993; Dorier & Sierpenska, 2001; Dorier, Robert, Robinet & Rogalski, 2000; Stewart & Thomas, 2009). Across the literature on the teaching and learning of eigenvalues and eigenvectors, procedural thought processes are featured prominently. For example, Stewart and Thomas (2006) highlighted the example of the conceptual processes and difficulties students find in learning about eigenvalues and eigenvectors, where a formal definition may be immediately linked to a symbolic presentation and its manipulation. Thomas & Stewart (2011) highlighted a difficulty students find when faced with formal definitions for eigenvalues and eigenvectors. Since these definitions contain an embedded symbolic form ($Ax = \lambda x$), students often move quickly into symbolic manipulations of algebraic and matrix representations such as transforming $Ax = \lambda x$ to $(A - \lambda I)x = 0$ without making sense of the reasons behind these symbolic shifts. Schonefeld (1995) used eigenpictures ("stroboscopic" pictures) to show x and Ax at the same time by using multiple line segments on the x-y-axis. He observed that graphical representations of eigenvalues and eigenvectors got little attention in the literature and that a picture may benefit more than algebraic presentations. It is also documented that students struggle to coordinate algebraic with geometric interpretations (e.g. Stewart & Thomas, 2010; Larson & Zandieh, 2013) and the students' understanding of eigenvectors is not always well connected to concepts of other topics of linear algebra (Lapp, Nyman, & Berry, 2010). To support students in developing a better understanding of the formal definition and the geometrical interpretations of the eigenvalues and eigenvectors, researchers have developed a variety of instructional interventions (e.g. Tabaghi & Sinclair, 2013; Zanidieh, Wawro, & Rasmusen, 2016).

Researchers often make reference to conceptual understanding and procedural understanding when discussing student's thinking about mathematical concepts (Hiebert, 1986). To operationalize our distinction between concepts and procedures, we draw on Vinner's (1997) distinction between conceptual and analytical behavior. According to Vinner (1997), students are in a conceptual mode of thinking if their behavior provides evidence that they are attending to concepts, their meanings and their interrelations. On the other hand, students are in an analytical mode of thinking when solving routine mathematical problems if they act in the way expected and certain analytical thought processes occur. In case they do not act in such ways but succeed in making the impression that they are analytically involved in problem solving then they are in pseudo-analytical mode of thinking. Vinner (1997) argued that in most problem solving situations when students are asked to solve a problem their focus usually is on which procedure should be chosen and not on why a certain procedure works. Based on this, we draw on Vinner's (1997) definition of conceptual versus analytical (procedural) as an analytic tool for interpreting student's answers to the questions involving eigenvalues and eigenvectors.

Data Sources

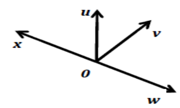
In our previous work, we have developed an assessment that covers the four focal topics of linear algebra mentioned earlier (Haider et al. 2016). This assessment was administered at the end of linear algebra course at different public and private institutions across the country. This was a paper-and-pencil assessment that includes 9 items and the students were allocated one hour to complete. It was designed to measure students' understanding of introductory linear algebra topics. Every item of the assessment contained a component of open-ended justification for students to elaborate their conceptual understanding of the topic.

Figure 1. Question # 8 and 9 (Eigenvalues/Eigenvectors) of the assessment

8. Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$? Why or why not?

9. Suppose the vector x is a real-valued eigenvector of the matrix M and that the entries of M are also real-valued.

   a. What could be the result of the product Mx? (Check all that apply.)



- ☐ i: Mx could be u
- ☐ ii: Mx could be v
- ☐ iii: Mx could be w
- ☐ iv: Mx could be **0**
- ☐ v: Mx could be x
- ☐ vi: None of the above

   b. Explain your reasoning for your choice(s) in part a.

We have collected assessment data from two groups of students: students whose instructors received instructional supports to teach linear algebra using a particular inquiry-oriented approach as part of the NSF-supported TIMES research project (who we will refer to as TIMES students), and students whose instructors did not receive these supports (who we will refer to as non-TIMES students. The instructors who participated in the TIMES project attended a 3-day summer workshop, participated online workgroup conversations for one hour per week for a semester, and implemented inquiry-oriented curricular material in their linear algebra class. We have collected the assessment data of 126 Times students across six TIMES instructors and 129 non-Times students across three non-TIMES instructors from different institutions in the US. Non-TIMES instructors were recruited from linear algebra instructors either at the same institutions as TIMES instructors or at other similar institutions (e.g. similar geographic area, similar size of student population, similar acceptance rate at institution) to collect assessment data for comparison of TIMES and non-TIMES students. In this study, we focused on in-depth analysis of students' reasoning on the assessment questions related to eigenvalues and eigenvectors. Both items are shown in the figures above.

The inquiry-oriented approach to learning eigenvalues and eigenvectors associated with this study is characterized in detail elsewhere (Zandieh, Wawro & Rasmussen, 2016). This approach supports students in coming to first learn about eigenvalues and eigenvectors as a set of "stretch" factors and directions that can be used to more easily characterize a geometric transformation. In this sequence of tasks, students first work to describe the image of a figure in a plane under a transformation that is easily described in a non-standard coordinate system. Students then work to label points using standard and non-standard coordinate systems corresponding to the previous task; they also find matrices that transform points from one coordinate system to the other. The instructor works to link this work to the matrix equation $A=PDP^{-1}$ and subsequent tasks aim to leverage this conceptual basis as students learn more traditional computational methods associated with computing eigenvalues and eigenvectors.

Methods of Analysis

To identify different types of students' approaches to eigenvectors and eigenvalues assessment items, we conducted our analysis in three stages: (I) developed a coding scheme for both the descriptive (open ended) part and the non-descriptive (multiple-choice) part of the items, (II) coded assessment data by following the coding scheme, (III) ran statistical analysis of coded data for descriptive statistics and t-test to compare the performance of TIMES and non-TIMES students.

At the first stage, we decide the coding scheme for non-descriptive part of the problems. Later, we looked into the descriptive parts of the problems. We identified different correct students' approaches for both problems, which helped us to refine our coding scheme. The coding scheme for question 8 and 9 is given in the table below:

	Points Awarded and Criteria	Comments
Question 8	3 Points: For any of the following answers. 1: $\det(A-\lambda I)=0$ implies $(x-1)(x-4)=0$ implies $x=1$ or $x=4$ implies $\lambda=2$ is not an eigenvalue for the matrix A. 2: $\det(A-2I)=-2\neq 0$ implies $\lambda=2$ is not an eigenvalue of A. 3: $(A-2I)\begin{bmatrix}x\\y\end{bmatrix}=0$ implies $x=0$ and $y=0$ which is the trivial vector, implies $\lambda=2$ is not an eigenvector for the matrix A.	Fully correct answer
	2 Points: if any answer with computational mistakes and with good justification and conclusion will be credited with 2 credits (computational mistaken attempt cannot be awarded full credit like a fully correct answer)	Partially correct answer
	1 Point: if any answer missing any of the three rules (correct answer, correct computation and correct explanation) will be missing credits depending on the number of rules missed.	Some procedural knowledge
Question 9	3 Points: Writing $Mx=\lambda x$ or mentioning Mx as a constant multiple of x/scalar multiple of x with or without mentioning what the values of this constant/scalar; 0, 1 and -1 and giving the options x, w, and 0.	Fully correct answer
	2 Points: If a student mentions $Mx=\lambda x$ and miss any of the possible values of constant/scalar; 0, 1, and -1 and/or miss one of the given options x w, and 0.	Partially correct answer
	1 Point: Writing $Mx=\lambda x$ and/or miss two of the possible values of constant/scalar; 0, 1, and -1 and/or miss two of the given options x, w and 0.	Some understanding

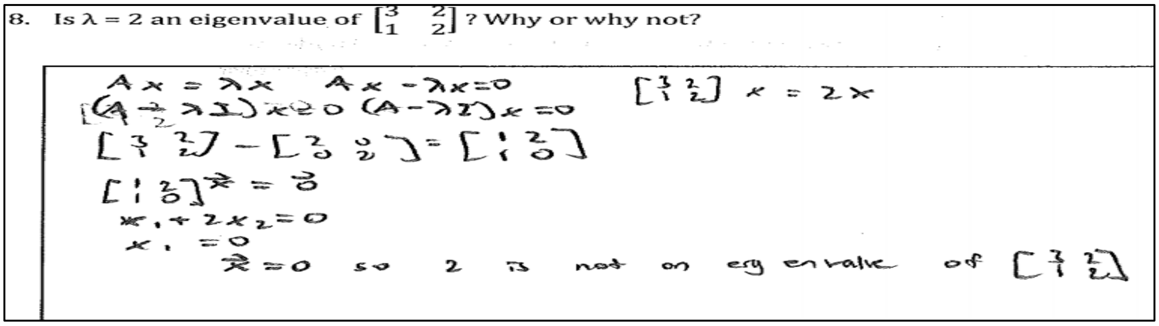
Table 1. Criteria for Awarding Points for question 8 and 9

At the second stage of analysis, we coded assessment data by following the above grading scheme. Two researchers looked at every students' attempt and decide a score independently and then matched with each other. If both researchers assigned different score to a particular student, then they discussed according to the grading scheme and agreed on a common score for that student. If both researchers have disagreement about a particular score, then a third researcher was consulted to make a consensus. At the third stage of statistical analysis, we checked the descriptive statistics to see the overall performance of students on the eigenvalue and eigenvectors questions. We also compared TIMES students with Non-TIMES students for both questions. We used t-test to compare the difference of means between questions 8 and 9 for both groups.

Initial Findings

In this section we summarize how students reasoned about the eigenvectors and eigenvalues items and how they med on both questions. We will provide some examples to show how they reasoned then point out using the quantitative data how the TIMES students performed compared to Non-Times students.

The following example shows that the student has a conceptual and procedural understanding in the sense that he/she by solving $(A - 2I)x = 0$, finding only the trivial solution and concluding that 2 is not an eigenvalue which is a nonstandard solution.


8. Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$? Why or why not?

$Ax = \lambda x$ $Ax - \lambda x = 0$ $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} x = 2x$

$(A - \lambda I)x = 0$

$\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \vec{x} = \vec{0}$

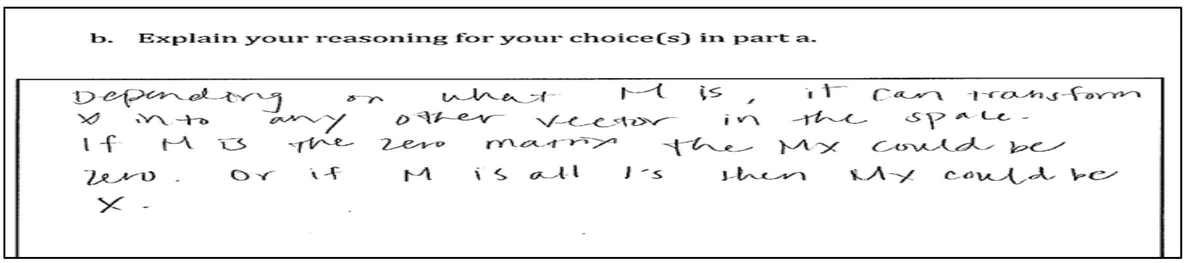
$x_1 + 2x_2 = 0$

$x_1 = 0$

$\vec{x} = 0$ so 2 is not an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

Figure 2: A student's example for Q8

Another example of a student who is interpreting the matrix M as something that can change in interpreting the eigenvector equation.


b. Explain your reasoning for your choice(s) in part a.

Depending on what M is, it can transform x into any other vector in the space. If M is the zero matrix the Mx could be zero. Or if M is all 1's then Mx could be x.

Figure 3: A student's example for Q9b

To see the difference in the students' performance we paid attention to the mean and standard deviation of the coded data. We also used t-test to compare the difference of means between both groups.

Question	All Students	TIMES Students	Non-TIMES Students
Q 8: Maximum Possible Points 3	Mean: 1.85 SD: 1.31	Mean: 2.0 SD: 1.23	Mean: 1.71 SD: 1.37
Q 9 (a & b): Maximum Possible Points 9	Mean: 1.50 SD: 0.82	Mean: 1.59 SD: 0.89	Mean: 1.42 SD: 0.74
Q 9 (b part only) Maximum Possible Points 3		Mean: 1.59	Mean: 0.54
		p-value = 0.0000273 < 0.05	

Table 2. Summary of TIMES and Non-TIMES Students' Performance

When examining the quantitative data to see how the two groups compared we noticed two things that seemed particularly noteworthy. First, both groups did better on the procedural item than the conceptual one. Second, TIMES students did better than non-TIMES on both items, but they outperformed non-TIMES on the conceptual item at a higher rate. In our session we will more deeply explore evidence of conceptual and procedural reasoning as it appeared in students' responses to the two items, and differences between the two groups of students.

References

Carlson, D. (1993). Teaching linear algebra: Must the fog always roll in*? The College Mathematics Journal, 24*(1), 29–40.

Dorier, J. L., Robert, A., Robinet, J., & Rogalski, M. (2000). On a research programme concerning the teaching and learning of linear algebra in the first-year of a French science university. *International Journal of Mathematical Education in Science and Technology*, *31*(1), 27-35.

Dorier, J. L., & Sierpinska, A. (2001). Research into the teaching and learning of linear algebra. In *The teaching and learning of mathematics at university level* (pp. 255-273). Springer Netherlands.

Haider, M., Bouhjar, K., Findley, K., Quea, R., Keegan, B., & Andrews-Larson, C. (2016). Using student reasoning to inform assessment development in linear algebra. In Tim Fukawa-Connelly, Nicole E. Infante, Megan Wawro, & Stacy Brown (Eds.), *19th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 163-177). Pittsburgh, PA.

Hiebert, J. (1986). Conceptual knowledge and procedural knowledge: The case of mathematics. *Hilsdale NJ: Lawrence Erlbaum Associates*.

Lapp, D. A., Nyman, M. A., & Berry, J. S. (2010). Student connections of linear algebra concepts: an analysis of concept maps. *International Journal of Mathematical Education in Science and Technology*, *41*(1), 1-18.

Larson, C. & Zandieh, M. (2013). Three interpretations of the matrix equation Ax=b. *For the Learning of Mathematics, 33*(2)*,* 11-17.

Schonefeld, S. (1995). Eigenpictures: picturing the eigenvector problem. *The College Mathematics Journal*, *26*(4), 316-319.

Stewart, S., and Thomas, M. (2006). "Process-object difficulties in linear algebra: Eigenvalues and eigenvectors." *International Group for the Psychology of Mathematics Education*: 185.

Stewart, S. & Thomas, M. (2009). A framework for mathematical thinking: The case of linear algebra. *International Journal of Mathematical Education in Science and Technology, 40*(7), 951-961.

Stewart, S., & Thomas, M. O. (2010). Student learning of basis, span and linear independence in linear algebra. *International Journal of Mathematical Education in Science and Technology*, *41*(2), 173-188.

Thomas, M., and Stewart, S. (2011). "Eigenvalues and eigenvectors: Embodied, symbolic and formal thinking." *Mathematics Education Research Journal* 23.3: 275-296.

Tabaghi, S. G., & Sinclair, N. (2013). Using dynamic geometry software to explore eigenvectors: The emergence of dynamic-synthetic-geometric thinking. *Technology, Knowledge and Learning*, *18*(3), 149-164.

Vinner, S. (1997). "The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning." Educational Studies in Mathematics 34.2: 97-129.

Wawro, M., Rasmussen, C., Zandieh, M., & Larson, C. (2013). Design research within undergraduate mathematics education: An example from introductory linear algebra. *Educational design research—Part B: Illustrative cases,* 905-925.

Zandieh, M., Wawro, M., & Rasmussen, C. (presented 2016, February). Symbolizing and Brokering in an Inquiry Oriented Linear Algebra Classroom. *Paper presented at Research on Undergraduate Mathematics Education, SIGMAA on RUME*, Pittsburgh, PA.