Common Algebraic Errors in Calculus Courses

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College mathematics instructors often view the final problem solving steps in their respective disciplines as “just Algebra”, but in reality, a weak foundation in Algebra may be the cause of failure for many college students. The purpose of this paper is to identify common algebraic errors students make in college level mathematics courses that plague their ability to succeed in higher level courses. The identification of these common errors will aid in the creation of a model for intervention.

Keywords: algebra, common errors, calculus

Introduction

As early as 1910 common errors related to arithmetic and rational number computation and the difficulties students were facing in learning mathematics were noted by De Morgan. Since then other researchers have catalogued common errors in computation and algebra (Ashlock, 2010; Benander & Clement, 1985; Booth, Barbieri, Eyer & Paré-Blagoev, 2014). Benander and Clement (1985) catalogued errors students made in basic arithmetic and algebra courses. Their work involved classroom observations and resulted in 11 categories of common errors including, basic problem solving skills, averages, whole numbers, fractions, decimals, percents, integers, exponents, simple equations, ratios and proportions, geometry, and graphing. Ashlock (2010) focused on the mathematics work of school-aged children and on helping instructors thoughtfully analyze their students’ work in order to discover patterns in their errors for the purpose of improving instruction. Ashlock suggests that as students learn about mathematical operations and methods of computation, they often develop and adopt misconceptions and procedural errors. Teachers who understand that this occurs and are able to identify these problems in their students’ work can develop strategies to help students. In a more recent study, Booth, et al. (2014) focused on the errors in algebra with school-aged students and identified errors that were “persistent and pernicious” given their predictive ability for student difficulty on standardized test items. Their study involved an in-depth analysis of students’ errors during problem solving at different points during the year and resulted in the classification of these errors which include: variable errors, negative sign errors, equality/inequality errors, operation errors, mathematical properties errors, and fraction errors.

Drouhard and Teppo’s (2004) work presented the idea of denotation and suggests that it is a developed sense about what one is writing and a lack of sense regarding denotation creates significant problems for students. They note “that students with poor capabilities to recognize this aspect of the meaning of an expression often make endless calculations because they do not know in what direction to go and when to stop” (p. 235). Harel, Fuller, and Rabin (2008) further
comment on the idea of meaning and denotation indicating that students often cancel within problems without attending to the quantitative meaning of their action. For example, Harel et al. (2008) states “it is not uncommon for students to manipulate symbols without a meaningful basis that is grounded in the context in which the symbols arise; for instance, a student might write: 
\[
\frac{\log a + \log b}{\log c} = \frac{a + b}{c}
\] (p. 116). In this case, students may be overgeneralizing their use of the distributive property and cancel “log” without considering the quantitative meaning of their action. Harel (2007) suggests that the lack of emphasis on mathematical meaning that students, and perhaps their teachers, apply to mathematical symbols creates what is referred to as a *non-referential symbolic* way of thinking and that this way of thinking can be tied to a myriad of algebra errors. Sfard and Linchevski (1994) believe that students must be motivated “to actively struggle for meaning at every stage of learning” (p. 225). They are concerned that “if not challenged, the pupil may soon reach the point of no return, beyond which what is acceptable only as a temporary way of looking at things will freeze into a permanent perspective” (p. 225).

Mason (2002) in his framework: *Manipulating-Getting-a-sense-of-Articulating*, emphasizes that students must be given opportunities to make sense of situations. He believes that “students want, indeed need, confidence-inspiring familiar objects to manipulate and on which to try out new ideas so that they can literally ‘make sense’ of them” (p. 187). In Harel and Sowder’s (2005) opinion “instruction (or curriculum) that ignores sense-making, for example, can scarcely be expected to produce sense-making students” (p. 46).

Although, research on students’ difficulties with algebra in school has been well documented (e.g. Kieran, 1992; Hoch & Dreyfus, 2004; Stacey, Chick, & Kendal, 2004), studies on occurrence of these errors in college level mathematics courses is scarce. In response to this need a pilot study conducted by Stewart and Reeder (2017) revealed how the unresolved high school algebra misconceptions and shortcomings may create major complications in college mathematics courses. Figure 1 demonstrates the progress and complexities of mathematical ideas as students approach the calculus courses (Stewart, 2017, p. vii).

\[
\begin{align*}
\frac{1}{2} + \frac{1}{16} & = 50 \\
\frac{1}{x^2} + \frac{1}{x^4} & \\
\frac{d}{dx} \left( \frac{x^2}{16} \right) & \\
\lim_{x \to 0} \frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{x} &
\end{align*}
\]

Figure 1. The unresolved issues with fractions compound as students confront limits in calculus.

While it appears that many students follow the theories that are introduced in calculus courses, in many cases not having a rich algebra background prevents students from completing basic tasks. Many students become particularly frustrated as they realize that the fast pace of college mathematics lectures and new material are not going to wait for them to catch up. On the other
hand, the instructors become disappointed with students’ performances as they note instances of algebra errors that should have been resolved years ago in high school. The aim of this study was to identify and categorize students’ common algebra errors in entry university mathematics courses. This paper does not deal with the question of why students are making such errors and continue to make them, but rather seeks to identify the type of errors that are most commonly made.

Method
Our research team included two mathematics educators working with three graduate students and a cognitive psychologist who specializes in children’s algebra thinking process as our consultant. Informed by the previous work and findings from their pilot study, this research team collected data from entry level college mathematics courses from a university in the Southwest United States. Data were gathered from approximately 600 students’ final exams and tests from the following mathematics courses: College Algebra; Pre-Calculus for Business, and Calculus and Analytical Geometry I during a single semester. For the purposes of this study, only the results from the Pre-Calculus for Business course will be discussed. Given the data for this research project were exams for actual college mathematics courses, the data had ecological validity.

While the use of actual test and exam questions provided validity to the study in that participants were doing the mathematics they are and will be asked to do in their college level mathematics courses rather than working through problems developed for the purpose of the study that might invoke or invite certain errors, this created challenges for coding and predicting the kinds of errors students might make. The ecological nature of the data called for a process of open coding and examination of codes and coding again.

As in the case of Booth et al. (2014) we anticipated that the errors would be concept driven. For example, certain errors were present at the beginning of the semester and did not appear again until near the end of the semester. However, we were interested to see if the errors persisted and showed up again in the final exam. Hence, we collected data from students’ tests 1, 2, 3 and the final examination, de-identified students’ names and gave each student a number as well as their instructor’s name in order to monitor their progress. For example, Jane Smith from Mr. Thomas’s class was coded as: “SP16 Course Number Thom 1”.

The data were scanned, and organized in a shared Dropbox for easy access by the members of the research team. The team met to review the findings of the pilot study and to discuss possible codes for common errors in student work. In order to ensure that the research team could work effectively to identify student errors, one set of exam questions was assigned to the group to analyze independently. The research team met the following week to discuss the coding categories and themes that emerged in an effort to determine an initial list of codes. This resulted in a list of the first nine codes Table 1. Further, the team agreed that beyond the initial nine codes, each person would add codes as needed. The data were then assigned to five members of the research team such that each set of test questions were analyzed independently by two
researchers. Each exam question was analyzed independently and coded using the nine pre-established codes combined with a process of open-coding. Following this analysis and coding, the team met to discuss the codes and establish what word would represent the kinds of errors that were emergent in the data (Rossman & Rallis, 2012) and what was collectively understood when that code was used. This discussion resulted in the addition of several new codes and the further definition of some of the codes already in use. Following this discussion, the team once again analyzed the data to re-code where necessary and open code as needed. This process resulted in the coding list presented in Table 1. When the coding process was complete, 4,328 test and final exam questions had been analyzed from students (N=163) in two instructors’ sections of Pre-Calculus for Business. Table 1 represents the percentage of each of those identified errors by code or error type. The results revealed that Simplifying, Sign Errors, Log Properties followed by Distributing, Isolating Variable and Exponents were the six highest percentages of the errors.

**Table 1. Percentage of errors by coding type.**

<table>
<thead>
<tr>
<th>Code and Description</th>
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<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Isolating Variable (Balance point) - Students are unable to correctly work with variables on both sides of an equation.</td>
<td>9.09</td>
<td>11. Substitution - Students substitute variables or values incorrectly,</td>
<td>1.67</td>
</tr>
<tr>
<td>2. Simplifying - Students are unable to simplify or do not simplify when needed.</td>
<td>12.86</td>
<td>12. Absolute Value - Student do not interpret absolute value correctly.</td>
<td>0.12</td>
</tr>
<tr>
<td>3. Exponents - Mistakes are made with exponents.</td>
<td>9.03</td>
<td>13. Function Notation - Students do not interpret function notation correctly.</td>
<td>4.08</td>
</tr>
<tr>
<td>4. Sign Errors - An error made with signs.</td>
<td>10.76</td>
<td>14. Mystery Zero - Students replace a variable with zero.</td>
<td>1.61</td>
</tr>
<tr>
<td>5. Fractions - Mistakes with computations with fractions or with working with variables within fraction notation.</td>
<td>1.55</td>
<td>15. Quadratic Equation - Unable to solve a problem using the quadratic equation.</td>
<td>1.55</td>
</tr>
<tr>
<td>6. Distributing - Misuse of or ignoring the distributive property.</td>
<td>9.96</td>
<td>16. Computational Error - Simple addition or other computation mistake.</td>
<td>1.36</td>
</tr>
<tr>
<td>7. Cancelling - Cancelling when it is not appropriate.</td>
<td>3.83</td>
<td>17. ln/e Conversion - Students convert between logarithmic and exponential forms of an equation incorrectly.</td>
<td>2.97</td>
</tr>
<tr>
<td>8. Radicals - Misuse of the radical sign or inability to convert the radical sign to an exponent representation.</td>
<td>8.97</td>
<td>18. Log properties - Students incorrectly combine or expand logarithmic properties.</td>
<td>10.95</td>
</tr>
</tbody>
</table>
The followings (see Table 2) are a sample of questions from tests 1, 2, 3 and the final exam. The questions were unified across all the sections and designed by the course coordinator. The tests and final exams contained multi-choice questions as well as long-answer questions. For the purpose of finding the algebraic errors, we only considered the long-answer questions.

Table 2. Sample Questions from Tests/Finals.

<table>
<thead>
<tr>
<th>Tests &amp; Final</th>
<th>Sample Questions</th>
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</table>
| T1           | 1) Solve the formula: \( y = \frac{T - mx}{k} - 12 \) for the variable \( x \).  
               2) Factor completely: \( 2a^2x^3 - 3a^2x^2 - 18x + 27 \)  
               4) Find the complete solution set for: \( \frac{1}{x^4} + \frac{1}{x^2} = 90 \) |
| T2           | 1) If the average rate of change of \( f(x) = x^2 + 3x + 7 \) from \( x = 2 \) to \( x = k \) is equal to 13, then find the value of \( k \)  
               2) If \( f(x) = 8 - 9x - 10x^2 \), then find and simplify \( \frac{f(x+h) - f(x)}{h} \) |
| T3           | 1) If you place \$100,000\) into an account drawing 7.5% interest compounded continuously, then how many years would it take to have \$350,000\) in that account?  
               \[ \text{round to the nearest year} \]  
               \[ A = P \left( 1 + \frac{r}{n} \right)^n \]  
               \[ A = Pe^{rt} \] |
| Final        | 8) Find the complete solution set for this equation: \( \log_3(x - 5) + \log_3(x + 3) = 2 \) |

Results
The result of this research show that college students carry with them misunderstandings and challenges related to algebra from their high school years into their entry level college mathematics courses. The types of errors they made varied and yet making errors was persistent and plagued the students’ abilities to learn new mathematics concepts throughout the semester from Test 1 to the Final examination. This study revealed that while algebra related errors are
evident in student work throughout the semester, the type of errors made are often dependent on the type of mathematics problems they are asked to solve. For example, errors with fractions may appear to be resolved as the semester moves along but it may well be that students are simply not asked to work problems that involve fractions near the end of the semester. Figures 2 and 3 below are examples of some of the error types resultant from this study. Figure 2 highlights the kind of error that was coded as Isolating Variable given that the student struggled to isolate the variable on one side of the equation. Nearly 10% of the errors found in this study were of this type. Figure 3 below provides an example of a students’ error with function notation.

Figure 2. Isolating Variables (Balance Point).

Figure 3. Function Notation.

**Discussion and Concluding Remarks**

This study investigated test and exam questions performed by 163 Pre-Calculus for Business students, and examined more than 4000 problems, in order to categorize the most common types of algebraic errors that college mathematics students make. Although, it is perceived that the type of exam questions posed, invited certain types of errors, the frequency of errors is persistent
throughout the semester regardless of problem type. The students consistently make mistakes and errors with algebra from Test 1 to the final examination.

While analyzing the data, we witnessed over and over again how algebra errors rapidly terminated the flow of the problem solving sequence for many students and resulted in incorrect solutions or no solutions. Many mathematics instructors believe that it is not our responsibility to teach or re-teach school algebra in college level courses. Realistically, going over algebra misconceptions is not a possibility and we have no time to repair them. Many of the errors found in this study reflect Drouhard et al.’s (2004) finding that students make endless calculations when they do not know what direction to go and Harel’s (2007) point about non-referential symbolic way of thinking. Booth et al. (2014) suggests that “the misconceptions underlying specific persistent errors are not corrected through typical instruction and may require additional intervention in order for students to learn correct strategies.” (p. 21).

Future research
Although, it is believed among many instructors that these types of algebraic errors should have been resolved years ago in high school and maybe nothing can be done at this stage, we believe that pinpointing the type of errors will help in creating interventions that remedy the algebraic errors. We continue to refine our common error types as more data become available from College Algebra and Calculus and Analytical Geometry I courses. Our next steps are to interview the instructors who taught these courses and seek more information from them in order to create the most effective interventions to help future calculus students.

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References


