Transitioning to Proof with Worked Examples

Dimitrios Papadopoulos
Drexel University

In this study, I explored the use of a worked-examples-based proof-writing framework as a pedagogical tool to improve undergraduate students’ ability to construct proofs. Over the course of three months, I ran a series of three workshops with five undergraduate students who had no prior experience with formal mathematical proof. In each workshop, participants worked through worksheets containing completed worked examples of mathematical proofs, followed by partially completed worked examples of proofs (to be completed by the participants), and, lastly, exercises. I collected and coded participants’ written work and reflections and explored changes in student proof-writing across workshop sessions. In this paper, I describe themes across student work and provide qualitative data supporting the benefits of incorporating the use of such a worked-examples-based proof-writing framework when introducing students to mathematical proof.

Key words: proof, worked examples, proof-writing framework

This study is motivated by the ongoing difficulties that students have in learning mathematical proof. As Moore (1994) points out, the transition from computational/procedural courses, such as calculus and differential equations, to proof-based courses is particularly difficult for undergraduate students. These difficulties are myriad and complex, and based on the experiences of both students and faculty, the traditional lecture-based classroom falls short of helping students overcome these difficulties. Fortunately, there is a growing body of knowledge on alternative methods of instruction such as guided reinvention, flipped classrooms, and worked examples. The purpose of this study is to explore the use of a worked-examples-based proof-writing framework as a pedagogical tool, implemented in a series of introductory workshops on mathematical proof-writing, to improve students’ ability to construct proofs.

Literature Review

The instructional practice at the center of this research is that of worked examples. In mathematics education, the word “example” has several interpretations. The word example can refer to a particular instance of a concept (e.g. 7 is an example of a prime number). Alternatively, the word example can refer to the demonstration of technique as the written solution to a particular problem/exercise, a worked-out example – or simply worked example; worked examples are typical of undergraduate mathematics textbooks (Weber, Porter, & Housman, 2008). Lithner (2003) found that in procedure-oriented courses students “almost always” make use of worked examples in completing their homework and that this approach was used by students of varying mathematical ability. Other recent studies suggest that undergraduate students in proof-based courses might use worked examples to inform their construction of proofs (Weber, 2004). While Lithner (2003) expressed concern that the use of worked examples leads to students completing homework assignments without developing conceptual understanding, cognitive psychologists have emphasized the merits of having students use worked examples in problem solving (e.g., Zhu & Simon, 1987; Atkinson, Derry, Rankl, & Worthing, 2000).
As pedagogy, the *worked examples* model has, in recent years, received considerable attention from researchers and educators, especially in mathematics, physics, and computer science. While the defining features of the *worked examples* pedagogical model are subject to debate, the common thread is the goal of providing the novice (student) with an expert’s problem solving framework, which is to be emulated in the interest of gaining conceptual understanding (Atkinson, Derry, Renkl, & Wortham, 2000). Three factors moderate the effectiveness of teaching by worked examples (Atkinson et al., 2000):

- Intra-example features – e.g. use of multiple modalities, clarity of substructure goals, completeness/incompleteness
- Inter-example features – e.g. example’s proximity to matched problems, multiple examples per problem type
- Students’ individual differences in processing examples – e.g. social incentives, self-explanation

In this study, I made use of findings in the *worked examples* literature to develop a proof-writing framework as well as a methodology for using that framework in a series of introductory workshops on mathematical proof-writing. In doing so, I took into consideration the inter- and intra-example features of successful *worked-examples-based* pedagogical models to create worksheets that deconstructed the proof-writing process according to that framework. This framework - that is, the specific way in which proof-writing is broken down into a series of subgoals - will be referred to as a *worked-examples-based proof-writing framework*.

**Methods**

**Participants.** The participants for this study were five second-year undergraduate students majoring in engineering, all with the very similar mathematical backgrounds. All of them had completed the first-year calculus sequence and were enrolled in an ordinary differential equations course at the time of the workshops. None of these students reported having any experience with mathematical proof, beyond high school geometry.

**Materials.** The theorems and exercises used in the workshops were drawn from introductory texts on mathematical proof-writing to reflect the material one would typically encounter in a transition to proofs course. For each session, worksheets were prepared which contained introductory information, such as definitions and descriptions of proofs types, fully worked-out examples, partially worked-out examples, and exercises.

**Structure of Worked Examples.** The structure of the worked examples was intended to decrease the cognitive load of the proof-writing process by encouraging students to deal with the various aspects of proof, as defined by Selden and Selden (2009), separately. These aspects of proof form the “conceptually meaningful chunks” or subgoals of the worked examples. The structure outline is as follows:

1. Proof construction
   a. Breakdown of problem/theorem statement into hypothesis and conclusion
   b. Brainstorming
      i. Identifying relevant and related definitions, axioms, and theorems
      ii. Identifying the end goal by considering the next-to-last step of the proof
   c. Translating the problem/theorem statement into hypothesis and conclusion into appropriate mathematical terminology and notation
   d. Application of definitions, axioms, and theorems to hypothesis
As Catrambone (1994, 1996) argues, explicitly labeling subgoals improves the effectiveness of the worked examples. Furthermore, the subgoals of part (1) – the proof construction - are expected to elicit directed self-explanation from students. Part (1) of the worked examples corresponds primarily to Hierarchical Structure, Construction Path, and the Problem-Centered part of the proof, while part (2) – the formal write-up of the proof - corresponds primarily to the Formal-Rhetorical and Proof Framework parts of the proof.

Hierarchical Structure refers to “knowing what the proof has to accomplish and coordinating any sub-proofs or constructions” (Fukawa-Connelly, 2012, p. 328). By rewriting the problem/theorem statement in if-then form (assuming the original statement has not been presented in this way), the worked example identifies what the proof has to accomplish (conclusion) and the conditions under which this must be accomplished (hypothesis). For example, in Worked Example 1, the statement “Prove that the sum of two odd integers is even” is rewritten as “If two integers are odd, then their sum is even.”

The brainstorming part of the worked examples addressed the Hierarchical Structure and the Problem-Centered part of the proof. Here, the goal of the proof is further clarified, key ideas are determined, and “the ‘right’ resources” are brought to mind. In Worked Example 1, the definitions of an odd integer and an even integer are called upon. The next-to-last step (x + y = 2N) is also written, further clarifying the goal of the proof.

In translating the proof statement into appropriate mathematical notation and terminology, the worked example introduces the Formal-Rhetorical part of the proof. This requires “primarily behavioral knowledge to complete” (Fukawa-Connelly, 2012). In worked example 1, the statement “If two integers are odd, then their sum is even” is rewritten as

If \( x, y \in \mathbb{Z} \), then \( x + y \) is even.

The Application of definitions, axioms, and theorems to the hypothesis addresses the Hierarchical Structure and Formal-Rhetorical parts by “coordinating any sub-proofs or constructions” and “coordinating aspects of the proof”(Fukawa-Connelly, 2005).

The experiment phase of the worked example addresses the Construction Path and Problem-Centered parts of the proof. Here, a viable construction path is sought by exploring logical and algebraic connections between all aforementioned definitions, axioms, and theorems. During the presentation of this portion of the worked examples, I encouraged students to practice self-explanation.

Finally, having established a viable construction path, the formal write-up/presentation of the proof is focused on Formal-Rhetorical, Construction Path, and Proof Framework parts of the proof.

Data Collection & Analysis. In an interpretive qualitative study such as this, data is typically collected through interviews, observations, and document analysis (Merriam, 2002). In place of conducting individual interviews with participants, in Sessions 2 and 3 students were prompted to write written reflections for each problem. I coded their work and reflections for emergent themes, first on a granular level, then grouping those codes according to the five aspects of proof. Data collection methods are summarized in the following table:
<table>
<thead>
<tr>
<th>Data Collection Method</th>
<th>Description</th>
</tr>
</thead>
</table>
| Written Reflections    | For the Partial Examples in Sessions 2 & 3 students were asked to provide written responses to the following questions:  
1. Why did you choose to prove this statement via direct proof/contraposition/contradiction?  
2. What parallels can you draw between this proof and the examples above?  
3. Which parts of our proof framework were most useful in working on this proof?  
4. If you were unable to complete this proof, what gave you difficulty? |
| Observations           | Each seminar session was audio recorded. Students were observed as they worked during each seminar. |
| Document Analysis      | All student written work and reflections were collected and scanned at the end of each session. |

## Results

There were several notable themes in the ways in which students’ proof writing changed over the course of the three workshop sessions. The most notable changes took place during the first session. Student work varied significantly on the Pretest, but, with only a small number of exceptions, no valid formal mathematical proofs were submitted by any of the students.

However, following the introduction of the work examples, all of the students were able to write at least one complete and valid proof on the Session 1 worksheet, with most doing so for more than one example/exercise. In the following table we see that the number of correct and mostly correct proofs increased significantly between the Pretest (PT) and Session 1, and that the number of incorrect proofs remained relatively low over the next two sessions. Here, “Mostly Correct” refers to proofs with minor errors or omissions, such as leaving variables undeclared and minor algebraic or arithmetic errors. Furthermore, while most of the proofs submitted on the Pretest were empirical or colloquial in nature, all of the proofs (or proof attempts) submitted after the introduction of the worked examples were deductive.

<table>
<thead>
<tr>
<th>Session</th>
<th>Correct</th>
<th>Mostly Correct</th>
<th>Incorrect</th>
<th>Incomplete</th>
<th>Empirical or Colloquial</th>
<th>Deductive Proof</th>
<th>Out of</th>
</tr>
</thead>
<tbody>
<tr>
<td>PT</td>
<td>1</td>
<td>1</td>
<td>20</td>
<td>2</td>
<td>17</td>
<td>7</td>
<td>24</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>9</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>9</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>18</td>
<td>18</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>0</td>
<td>0</td>
<td>19</td>
<td>19</td>
</tr>
</tbody>
</table>

Below are Rasa’s and Paul’s solutions for the same exercises on the Pretest and the Session 1 worksheet. The task in this case was to prove that for integers \(a, b, \) and \(c\), if \(a|b\) and \(b|c\), then \(a|(b+c)\).
Rasa’s proof on the pretest was primarily colloquial in nature, but her proof on the Session 1 packet was almost entirely correct, with the exception of undeclared variables, $m$ and $n$. Modeling the formal write-ups in the worked examples, her proof followed mathematical convections such as use of notation, separating and labeling algebraic work, and writing in complete sentences. Paul, who attempted to operationalize the definition of divides on the Pretest, also wrote a correct proof on the Session 1 worksheet. Like Rasa, he wrote in complete sentences, indented his algebraic work, and clearly stated his conclusion.

In addition to correctness, on a more granular level, the formal-rhetorical features of students’ proofs continued to evolve over the course of the three sessions. In particular, all of the students included conventional notation and language, and declared all primary and secondary variables in the partial examples in Session 1. Here, primary variables refer to those which are used to represent quantities referenced in the hypothesis/assumptions of a given statement to be proven, and secondary variables refer to those used to operationalize a definition. In most cases students became less consistent with declaring secondary variables in the later sessions. The following table compares the total number of variables declared by all students in each session with the minimum number of variables required to construct correct proofs on those exercises. To establish these minima, I wrote out proofs for each example and exercise and had several mathematicians look over that work to determine if I had used the minimum number of variables possible to construct valid proof:

<table>
<thead>
<tr>
<th>Example/Exercise</th>
<th>Primary Variables Total / Minimum</th>
<th>Secondary Variables Total / Minimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretest Summary</td>
<td>$27 / 40 = 67.5%$</td>
<td>$10 / 45 = 22.2%$</td>
</tr>
<tr>
<td>Session 1 Summary</td>
<td>$33 / 37 = 89.2%$</td>
<td>$33 / 45 = 73.3%$</td>
</tr>
<tr>
<td>Session 2 Summary</td>
<td>$17 / 20 = 85.0%$</td>
<td>$22 / 38 = 57.9%$</td>
</tr>
<tr>
<td>Session 3 Summary</td>
<td>$32 / 38 = 84.2%$</td>
<td>$25 / 48 = 52.1%$</td>
</tr>
</tbody>
</table>
On average, students made and maintained significant progress after the Pretest in declaring primary variables, from 67.5% declared on the Pretest to 89.2% on the Session 1 worksheet and then maintaining roughly 85% for the last two sessions. However, in declaring secondary variables, there was a significant increase from the Pretest to Session 1, but then a decline in the final two sessions.

Finally, by Session 3, most students had begun to demonstrate a strategic approach to proof-writing, engaging in brainstorming and experimenting prior to attempting to write formal proofs. Consider, for example, Heather’s work in Session 2 (left) and Rasa’s work from Session 3 (right):

![Contrapositive Proof - Letter][305x586]

While Heather was unable to write an entirely correct proof for any of the exercises in this session, she did experiment with several statements to determine which might be most easily solved as either direct proof or proof by the contrapositive. In doing so, she started each proof and made notes regarding what “might work” and whether to use contraposition or not. For 6a, she did in fact write a mostly correct proof, although once again, variables were not properly declared. Rasa began her work on Exercise 1 in Session 3 by rewriting the given claim according to the three proof methods covered in these workshops, prior to selecting proof-by-contradiction and attempting to construct a proof. The work of both students here indicates planning as a separate activity from the formal writing-up of a proof.

In their written reflections, students most frequently identified brainstorming and experimenting as being the most valuable features of the worked-examples-based proof-writing framework. The following examples highlight some of their reasons for saying so:

<table>
<thead>
<tr>
<th>Student</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nadia</td>
<td>Definitely brainstorming is very important to understand what we are trying to proof but experimenting was important to fully understand the process.</td>
</tr>
<tr>
<td>Heather</td>
<td>There is NO way to know if the statements work w/ out experimentation.</td>
</tr>
<tr>
<td>Rasa</td>
<td>Experimenting and thinking which process would work best beforehand helped the most.</td>
</tr>
</tbody>
</table>
Overall, these responses suggest that students found these steps useful because they allowed them to make sense of what the proof required prior to attempting a formal write-up.

Discussion

Most undergraduate mathematics students in the United States typically encounter formal mathematical proof-writing for the first time in a transition (or introduction) to proofs course. Such courses tend to following computation-centered courses, such as the typical freshman level calculus courses, and typically span only one semester. Because formal mathematical proof plays such a central role in advanced mathematics courses, it is important that we take full advantage of these transition courses to equip students with tools and experiences that will allow them to succeed in later proof-based courses. In this study, I aimed to explore the ways in which a worked-examples-based proof-writing framework could support students during this transitionary period.

This study was conducted independently of any transition to proofs course in order to focus on the ways in which this worked-examples-based proof-writing framework would affect novice students’ proof-writing. However, the implications of this study for instruction lie primarily in the potential use of this type of framework as a tool to supplement instruction in an introductory course to proof. Worked examples are by their nature supplementary and of primary value to the novice.

Lithner (2003, 2004) expressed concern that in relying too heavily on worked examples to solve problems, students sacrifice opportunities to build on their conceptual knowledge and problem-solving strategies. For this reason, this study treats worked examples as a means to “provide an expert’s problem-solving model for the learner to study and emulate” (Atkinson et al., 2000, pp. 181-182) and as a means to prompt guided exploration. As has been found in much of the worked examples literature, without examples to study, students may develop, and over time reinforce, novice strategies that ignore deeper structures in problem-solving activities (Weber et al., 2008).

The keys findings of this study suggest that a worked-examples based framework can both help novice students develop some of the basic proficiencies necessary for constructing and formally writing up mathematical proofs, and over time, facilitate productive habits, such as brainstorming and experimenting prior to attempting a proof write-up. These basic proficiencies include declaring variables, operationalizing definitions, using conventional mathematical language and notation, and structuring proofs in an organized, readable way. By encouraging brainstorming and experimenting, using this framework may help students develop some of the strategic knowledge required for mathematical proof-writing.

Regarding implementation, more research (see future research recommendations below) may be required to determine how one might best incorporate this framework into a standard transition to proofs course. However, given that the students who participated in this study received little instruction, and instead learned to construct proofs primarily by modeling worked examples, it may be that worksheets such as those used in this study could be used as supplementary materials in an introductory course on proof. Of course, they could be tailored to suit the specific content of the course and the needs of the instructor and students. The advantage to this type of purely supplemental use of the proof-writing framework is that little or no change is required of the instructor in terms of preparation or preferred teaching style.
References


