A Case Study in Constructing Set-based Meanings for Conditional Truth

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We present a case study of Hugo’s construction of Euler diagrams to develop set-based meanings for mathematical conditionals. This episode arose in a teaching experiment guiding students to reinvent mathematical logic from their reasoning about meaningful mathematical statements. We intended for Hugo to develop a subset meaning for conditional truth. Hugo successfully identified and used this condition, but he also introduced another formally equivalent meaning for conditional truth. We discuss the shifts in his thinking necessary for developing set-based reasoning and how this case influenced our goals for logic learning.

Keywords: logic, Euler diagrams, conditionals, teaching experiment

Though the modern formalizations of mathematical logic and language are relatively young (Frege, 1879; Russell, 1903), these accounts have become integral to the normative understanding of the language of proof-oriented mathematics. Azzouni (2006) argues that the essential novelty was developing a formal language in which meaning and truth are defined in purely syntactic ways. Because students must abide by these formal conventions of language to some degree, many transition to proof courses teach mathematical logic (Selden, 2012). However, the existing literature provides relatively little insight about logic instruction and the meanings that students must develop from that instruction (exceptions include Antonini, 2001; Bardelle, 2013; Barnard, 1995; Durand-Guerrier, 2003; Hawthorne & Rasmussen, 2014).

Regarding instruction, logic can be taught using everyday statements (e.g. Epp, 2003), formal syntax (e.g. Hawthorne & Rasmussen, 2014), or mathematical statements (e.g. Dubinsky & Yiparaki, 2000). Regarding student learning, there are various ways in which a student may affirm a conditional such as “If an integer \( x \) is a multiple of 6, then \( x \) is a multiple of 3”:

1. as an empirical generalization inducted from a series of examples,
2. based on properties such as the spacing of these multiples on the number line,
3. as the result of a proof (maybe implicitly) using the theorem “if \( a | b \) and \( b | c \), then \( a | c \),”
4. as a subset relation between the set of multiples of 6 and the set of multiples of 3, or
5. as not false because there is no multiple of 6 that is not a multiple of 3.

We know little about students’ meanings (Thompson, Carlson, Byerley, & Hatfield, 2013) for conditional truth, how they develop, and which should be privileged by logic instruction. We find it useful to distinguish between those meanings for conditional truth that intrinsically rely on the mathematical content of the sentence (examples 1-3) and those that rely on generalizable criteria (examples 3-5). We place 3 in both categories because the theorem used to prove is mathematically specific, but one can generalize the criterion that a conditional is true if there is a proof of the conclusions from the hypotheses (Weber & Alcock’s, 2005, warranted conditional). How and when is it important for students to develop generalizable meanings for conditional truth and even to become reflectively aware of such meanings?

The Current Investigation

In this report, we lay some foundation for investigating these questions through a detailed case study of one students’ learning about conditional truth and contrapositive equivalence. By
documenting particular students’ pathways of learning, we can discern important challenges and opportunities for instruction. This case comes from a series of teaching experiments attempting to guide undergraduate students to reinvent mathematical logic (Dawkins & Cook, 2015, in press). We taught logic by presenting students with meaningful mathematical statements all of the same logical form (disjunctions, conditionals, then multiply quantified). By letting students assign truth-values, reflect on their strategies, and construct negations, we intended for students to reinvent truth-functional logic as a formalization of their own reasoning and languaging.

We frame our findings in terms of student meanings, which Thompson et al. (2013) define as the set of inferences available to a student as a result of understanding something in a particular way. To illustrate this in our context, consider the tools available to a student for interpreting a mathematical statement and deciding its truth-value. While our participants exhibited various strategies for interpreting general statements, many times their reasoning focused either on examples, properties, or sets (which correspond respectively to examples 1, 2-3, and 4 above). By interpreting the sentence in terms of examples or properties, varying truth conditions and insights became available to students, which are parts of the meaning of the sentence for that student. We will thus discuss example-based, property-based, or set-based meanings. While students assigned statements their normative truth-values using all three, we found that reasoning with sets often afforded students the most fruitful strategies (Dawkins & Cook, 2016). Thus, in the experiment featured in this paper we attempted to guide students toward set-based truth conditions, which for conditional statements can be stated: The conditional “If for \( x \in S, P(x) \) then \( Q(x) \)” is true if and only if \( \{ x \in S | P(x) \} \subseteq \{ x \in S | Q(x) \} \).” We call this the subset meaning for conditional truth (or subset meaning for brevity). The primary contributions of this study are 1) documenting this student’s resources and challenges in developing the subset meaning and 2) documenting the novel meaning he created to affirm the contrapositive of a true conditional.

**Conceptual analysis of conditional truth**

We value Euler diagrams as a means of representing set relations relative to compound statements. The student featured in this report was familiar with similar diagrams from previous instruction (likely Venn diagrams), but he did not have fully normative meanings for how the diagram referred to mathematical objects. Dawkins and Cook (2015) point out at least one important meaning for understanding such diagrams as mathematicians do: the negation of a property corresponds to the complement of the set with the original property. Dawkins and Cook (2015) demonstrate that not all students associate the negation of a property with the complement set of examples, but such an understanding seems necessary for understanding why contrapositive conditionals have the same truth value (Figure 1).

<table>
<thead>
<tr>
<th>Original conditional</th>
<th>Contrapositive conditional</th>
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<td>“If for ( x \in S ) ( P(x) ), then ( Q(x) ).”</td>
<td>“If for ( x \in S ) not ( Q(x) ), then not ( P(x) ).”</td>
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<tr>
<td>is true whenever ( { x \in S</td>
<td>P(x) } \subseteq { x \in S</td>
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*Figure 1: Euler diagrams demonstrating why contrapositives have the same truth-value.*

Rather, students may associate the negation of a property with a proper subset of the complement – e.g. “not acute” means “obtuse,” or “not even” means “odd” – or they may associate the
negation with an overlapping property – e.g. “not a rectangle” means “is a parallelogram.” Such
semantic substitutions (Dawkins & Cook, 2016) do not afford the set structure that we intend for
students to develop. Students’ choices of semantic substitutes demonstrate a strong preference
for familiar categories precluding the Euler diagram’s novel partitions.

Background and Study Design
This study investigates Hugo’s learning during the time he and his partner, Elya, worked on
the set of conditional statements provided by the teacher/researcher (the second author). These
students were recruited from a Calculus 3 class at a mid-sized university in the United States, but
the experiment took place in one-hour sessions outside of class. The task progression for each
type of statement – disjunction, conditional, and multiply-quantified – included: 1) assigning
truth-values to all of the provided statements, 2) look for patterns in how they determined
whether the statements were true or false, 3) consider the sets of examples that made each
statement true and the sets that made them false, and 4) constructing negations. Hugo and Elya
spent 2.5 sessions studying mathematical disjunctions prior to working on conditionals. This
paper focuses on the subsequent 2.5 sessions they spent studying conditionals, especially the
second such session from which Elya happened to be absent. Table 1 presents some conditionals
they studied. After the students assigned truth-values and looked for patterns, the interviewer
asked Hugo and Elya to “Think about the set of all things that satisfy the if part and the set of all
things that satisfy the then part. And tell me about the relationship between those two.” This was
particularly intended to prompt students to formulate the subset meaning for conditional truth.

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<tr>
<td>1. If a number is a multiple of 3, then it is a multiple of 4.</td>
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<td>2. If a number is a multiple of 3, then it is a multiple of 6.</td>
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<tr>
<td>3. If a number is a multiple of 6, then it is a multiple of 3.</td>
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<tr>
<td>4. If a number is not a multiple of 6, then it is not a multiple of 3.</td>
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<tr>
<td>5. If a number is not a multiple of 3, then it is not a multiple of 6.</td>
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<tr>
<td>6. If a triangle is not acute, then it is obtuse.</td>
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<td>7. If a triangle is obtuse, then it is not acute.</td>
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<tr>
<td>8. If a triangle is not acute, then it is not equilateral.</td>
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<tr>
<td>9. If a quadrilateral is a rectangle, then it is a square.</td>
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<tr>
<td>10. If a quadrilateral is a rectangle, then it is a parallelogram.</td>
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<tr>
<td>11. If a quadrilateral is not a rhombus, then it is not a rectangle.</td>
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Table 1: Sample disjunctions that Hugo and Elya studied.

Consistent with teaching experiment methodology (Steffe & Thompson, 2000), the
teacher/researcher consistently formed second-order models of student understanding and tested
those models through subsequent questioning. These models were informed by findings from
previous studies (Dawkins & Cook, 2015, 2016), as were the learning goals articulated earlier in
the paper. The data analysis consisted of iterative analysis throughout the experiment and
retrospective analysis of the video recordings and artifacts of student work afterward. The
analysis presented in this paper focused particularly on the pair’s construction of set-based
meanings for conditionals, especially as facilitated by Euler diagrams. Retrospective analysis
similarly consisted of developing second-order models of Hugo’s understanding by forming and
testing hypotheses using the corpus of his mathematical activity. Dawkins and Cook (2015,
2016) present more thorough accounts of the teaching and data analysis methodologies.

Results
On Hugo and Elya’s first pass through the set of conditionals, they assigned to each the
normative truth-value. They quickly recognized that an example that satisfied the if part and not
the *then* part was a counterexample (their *counterexample criterion*). Based on their work with disjunctions, they understood that a single counterexample made a general statement false and they could articulate general conditions for declaring statements false. For instance, after the first five statements Hugo explained, “If we can come up with a case that fits the first [antecedent] but does not fit the second [consequent], then it [conditional] has to be false. We only need one case.” Even in the case of positive statements, Hugo focused on examples by populating Elya’s set-based explanations with particular examples. When she explained, “[Statement] Five would be true, because every multiple of 3—every multiple of 6 is a multiple of 3. So if it’s not a multiple of 3 it can’t be a multiple of 6,” Hugo elaborated, “Like 17, is not multiple of 3 but it is also not a multiple of 6.” Elya inferred statement 5 from statement 3, which we call a *contrapositive inference*, but her later work suggests that she did not understand contrapositive equivalence as a general property of conditionals. We hypothesize that her understanding of multiples supported her inference rather than reasoning about abstract logic or set relations. In addition to focusing on examples, Hugo also affirmed some conditionals using property relations. Regarding statement eight, he said, “Not acute would mean either a right triangle or an obtuse triangle. Neither of those can be equilateral so that would be true.”

After Elya and Hugo assessed all the provided conditionals, the interviewer asked them to look for patterns in the statements and why they were true or false. The pair noticed that many statements contained the same properties and identified the relationships commonly referred to as inverse (e.g. #3,4) and converse (e.g. #6,7) conditionals. After some discussion and conjectures about the truth-values of these related statements, the interviewer asked the pair to consider the sets related to each part of the statements. The conversation proceeded:

H(1): I’d say, if the statement is true then the set for the first part—I’m sorry the set of the second part will be included in the set of the first part.
I(2): Okay. Why do you say that?
H(3): Um, because if we said that it’s true then when we pick—something that’s true for the first part, then it has to be included in the second part for the whole statement to be true.
I(4): So its sounds to me like you’re saying there’s two possibilities. One is to say that the set—the *if* set can be sort of inside of—or contained in the *then* set. Or you can say the *then* set is contained in the *if* set. Which one do you—are claiming? So you’re saying, if the statement is true then what was the relationship here?
H(5): Then the—*then* will be inside *if*.
I(6): Okay. What do you think [Elya]?
E(7): I think the *if* has to be in the *then* but *then* doesn’t have to be in the *if*. ‘Cause there—when we looked at 3 there’s all the multiples of 6 are contained in multiples of 3 but all multiples of 3 are not contained in multiples of 6.
I(8): Do you agree with that, [Hugo]?
H(9): Yeah I think I’m good with that […]
I(10): What are thinking about in a particular example?
H(11): Uh, you wanna talk about [statement] 3. Um in like a circle, and multiples of 3—3,6,9,12. Um, multiples of 6 will be included in that circle. Like 6 and 12 are multiples of 6. So there’s an additional circle inside that includes some numbers but does not include others.
I(12): Okay, but sort of which are you calling the *if* part and which circle are you calling the *then*?
H(13): The *then* part would be the bigger one. The inside would be *then*—sorry other way around. *Then* is on the outside. *If* is in.
This appeared to be the first time Hugo reasoned about conditionals using sets, and, from the researcher perspective, his general explanation (turn 5) was inconsistent with his earlier semantic reasoning about the given statements. Once the interviewer pushed Hugo to explain his reasoning in a particular semantic context (turn 10), he recognized the subset meaning for conditional truth (turn 13) but even after this episode he sometimes returned to his intuition that the then set is contained in the if set. We conjecture that this intuition may be supported by the relationship between the properties in true conditionals. For instance, statement three is true because being a multiple of 6 means being a multiple of 2 and a multiple of 3, meaning the if property entails the then property. This explanation seems consistent with Hugo’s use of the word included in turns 1 and 3. Hugo’s population of the Euler diagram with numbers (Figure 2) appeared to serve as a bridge between his example-based meanings and the set reasoning the interviewer invited.

![Figure 2: Hugo’s Euler diagram for statement 6.](image)

Hugo and Elya began to use these Euler diagrams to explore the truth of the various conditionals provided. By the end of the first session on conditionals, Elya was able to produce an explanation for why two contrapositive statements were both true as portrayed in Figure 1, but Hugo showed little evidence of understanding her reasoning. The next session afforded Hugo opportunity to independently develop his understanding of conditional truth because Elya happened to be absent. To encourage rediscovery of contrapositive equivalence, the interviewer invited Hugo to explore multiple related statements — original, inverse, converse, and contrapositive — with the same Euler diagram. The dialogue proceeded:

I(14): So look at [statements] 2 through 5 then. ‘Cause 2 through 5 are gonna be—again—its a bunch of the same options but the same kinds of parts.

H(15): Two—the number is a multiple of 3. We got a circle—they’re all multiples of 3—3, 6, 9, 12. Then it is a multiple of 6. [draws a circle around the 6 and 12].

I(16): Okay so is this true or false? So if it’s a multiple of 3, then it’s a multiple of 6.

H(17): We said it was false because our set was this circle of multiples of 3, and then if you only want multiples of 6—you have this but you still have the 3 and the 9 which aren’t included in the little circle.

In turn 15, Hugo produced the diagram shown in Figure 3, labeling the outer circle “if 3x” and the inner circle “then 6x.” In turn 17, Hugo did not rely on his counterexample meaning, but rather declared statement 2 false because the sets failed to satisfy the subset condition (“aren’t included in the little circle”). When the interviewer asked him to consider statement 3 (the converse), he expected Hugo to refer to the same diagram. Instead Hugo began modifying the position of the circles around the example numbers. This suggested that Hugo’s meaning for the diagram did not merely reflect the invariant relationship between the multiples of 3 and 6. Instead, Hugo tied the circles’ meaning to the statement structure – if and then – such that shifting to the converse altered the diagram. The interviewer recreated the original diagram and encouraged Hugo to interpret statements 3-5 using the same diagram:

I(18): Okay, so we can sort of stick with the circles. But now what’s—so in the second statement, or now actually it’s number 3, what is it we’re saying?
H(19): If is a multiple of 6, we put the if here [switches if/then labels]. Then it’s a multiple of 3. So if we only talk about this, this set of numbers, does that included inside of 3x? Yes.
I(20): Yea, all the multiples of 6 here are multiples of 3. Okay, right. I agree, so this one is true.
H(21): So again, if the if is encased, enclosed, inside the bigger then circle, then it’s true. […]
I(22): I want you to try to use the same picture to talk about 4 and 5.
H(23): If it is not a multiple of 6, so—if its not in this circle, so we’re talking about outside the little circle. […] Does that make sense?—If not multiple of 6 then it is not a multiple of 3. So we’re talking about anything outside of that. Well 9 is still a multiple of 3, 15 is still multiple of 3. So that would be false.
I(24): Okay, where are the non-multiples of 3?
H(25): Non-multiples of 3 would be outside the little circle infinitely, so that would be—oh non multiples of 3. That would be—[marks diagram]—So if it is not a multiple of 6. So we said it’s outside the little circle. Then it is not multiple of 3, not multiple of 3 would be outside the bigger circle. So our set of numbers is inside the big circle and outside of this, but also outside the big circle.
I(26): Right, so what numbers are outside the big circle? Just give me a few examples.
H(27): 7, 11 […]
I(28): If it’s a non multiple of 6 then it is a non-multiple of 3.
H(29): It’s not a multiple of 6, like 3, 3 is multiple of 3 though.
I(30): Yea, so your counter-examples, like you said are these. ‘Cause they’re non-multiples of 6 that in fact are multiples of 3. Which, so I agree. [Statement] 4 is false for that reason. 3 is a counter-example. […]
H(31): So we have a ton of numbers that prove it yes but we have all the counterexamples would be inside the big circle and outside the little circle. [reads statement 5] So if we’re only talking about number that are not multiples of 3. So if anything outside the big circle—well our multiples of 6 are inside the circle, so if we talk about anything outside of the circle, obviously we’re not going to contain multiples of 6. So if our set of numbers are outside the big circle, then yeah, none of those are multiples of 6. So that would be true.

*Figure 3: Hugo’s diagram at turns 17 and 25*

Once Hugo recognized that he could use the same diagram to assess the converse (turn 19), he explicitly cited the subset meaning to affirm statement 3 (turn 21). The interviewer’s prompt to use the same diagram for statements 4 and 5 required Hugo to attend to the negation/complement relationship. In turn 23 he associated the complement of the inner circle with the non-multiples of 6, but he avoided coordinating the complement of both sets by identifying a counterexample (the criterion he used the previous day). In turns 25 and 31, Hugo recognized that his counterexamples exemplified a class of numbers represented by the space between the inner and outer circles. Thus while his identification of a counterexample built upon his previous day’s work, he now displayed a set-based meaning for counterexamples afforded by the Euler diagram.
When Hugo considered statement 5, he associated the first condition with the complement of the outer circle. However, instead of affirming this statement by the subset meaning as portrayed in Figure 1, Hugo argued that non-multiples of 3 (numbers outside the outer circle) could not be multiples of 6 (inside the inner circle). Stated another way, the intersection between the non-multiples of 3 and the multiples of 6 was empty. This empty intersection meaning for conditional truth again allowed Hugo to avoid coordinating two negations by parsing statement 5 as “if a numbers is a non-multiple of 3, then it is not a multiple of 6.”

The interviewer invited Hugo to consider the relationships among statements 2-5 in terms of converses and inverses. With interviewer prompting, Hugo acknowledged that since #4 was the inverse of #3 and #5 was the converse of #4, one could go from #3 to #5 by taking both the converse and inverse. Hugo said, “But if we have an if-then statement that’s true, we take the inverse and the switch [their term for converse]—we switch it and take the inverse, then—so far we’ve proved that it would be true. […] I’m observing [this pattern] but I’m trying to articulate why that is.” Thus, he observed that these statements shared the same truth-value and conjectured that this pattern might hold generally, but he was unable to justify why this occurred.

**Discussion**

Our initial goals in this section of the teaching experiment were for Hugo and Elya to develop the subset meaning for conditional truth and to use that meaning to justify contrapositive equivalence. Elya used set-based meanings spontaneously while Hugo needed to develop tools to move beyond his example and property-based meanings. We claim that Hugo’s Euler diagram, initially produced to record his example-based reasoning, served as a transformational record (Rasmussen & Marrongelle, 2006). It both allowed him to generalize his example-based meaning for counterexample and later allowed him to relate inverses, converses, and contrapositives.

Unlike prior study participants who struggled to associate negative properties with complement sets (Dawkins & Cook, 2015), Hugo developed this relationship in his interpretation of the diagram. However, he avoided coordinating two complements simultaneously for statements 4 and 5, as would be required by the subset meaning. He instead used counterexample and empty intersection meanings. While we could view this negatively as it falls short of adopting a purely syntactic and content general understanding of conditional truth – as one might desire in teaching formal logic – Hugo’s empty intersection meaning is logically equivalent to the subset meaning. Furthermore, he showed flexibility in parsing and interpreting the given statements, which could be fruitful in proof-oriented mathematical activity. We appreciate how the Euler diagram afforded Hugo valid mathematical inferences even if he could not justify those inferences, much as Elya’s semantic understanding supported contrapositive inferences. Hugo alternatively drew upon semantic information (about multiples or geometric shapes), linguistic competencies (parsing negative statements), and logical criteria (the subset meaning) to interpret the various statements provided, which we anticipate to be more consistent with proof-oriented reasoning than purely syntactic operations in a formal language.

This paper contributes to our understanding of students’ construction of logical structure in semantically rich settings and reveals hurdles and opportunities in students’ development of set-based meanings for compound statements. As was portrayed in the introduction, we prioritize students like Hugo and Elya developing generalizable meanings for mathematical truth. However, based on cases like Hugo’s we also value the flexibility to affirm the same statement in multiple, formally equivalent ways. Further study should continue to shed light on viable pathways for students to abstract their semantic reasoning into generalizable, syntactic tools and how these tools can be harnessed in students’ subsequent proof-oriented activity.
References


