This study reports three preservice secondary teachers’ abilities and tendencies to use representations in problem solving as well as their abilities to use realistic tasks after taking mathematics content and methods courses that emphasized the roles of representations and realistic tasks. Qualitative analyses showed that the preservice teachers developed beliefs that representations and realistic tasks are important components of secondary education and used motivational tasks in their instruction. However, they used the tasks mainly as the application of learned facts rather than as the departure of students’ construction of mathematical ideas. They also showed tendencies to use algebraic approaches in problem solving for grade 5-12 level tasks and had difficulties connecting algebraic and geometric representations when solving high school level algebra problems.

Keywords: Representations, Teacher Education, Mathematical Tasks, Problem Solving

Objectives

The importance of representations in mathematical teaching and learning has been emphasized in many studies (Brenner et al., 1997; Hiebert et al., 1997; Leinhardt, Zaslavsky & Stein, 1990; Thomson, 1994ab). Yet studies have suggested that many teachers have less than optimal understanding of the roles of representations and have difficulties using them in their instruction and in their own problem solving. Nevertheless, not much is known about how and in what ways teachers, especially secondary teachers, develop conceptions or knowledge of representations. This study addresses this under-documented area. It examines in what ways a series of mathematics and mathematics education courses that emphasized the roles of representations and tasks helped preservice secondary teachers to develop their mathematical knowledge for teaching. In particular, this study answers the following two research questions: (RQ 1) How were preservice secondary teachers’ tendencies and abilities to solve problems using representations after taking courses that emphasized the roles of representations and tasks? (RQ 2) How were preservice secondary teachers’ abilities to use tasks in instruction after taking courses that emphasized the roles of representations and tasks?

Framework and Literature Review

This study was conducted under the premises that knowledge and external representations are closely related and that mathematics is a human activity—“the activity of mathematicians that involves solving problems, looking for problems, and mathematizing subject matter” (Gravemeijer, 2008, p. 285). According to Hiebert and Carpenter (1992), “mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections” (p. 67). When internal representations are connected, the connections produce networks of knowledge, which are structured hierarchically or as a spider’s web. These internal representations and their connections, however, cannot be observed. Under the assumption that there is a relationship between external and internal representations, learners’ knowledge is often viewed through their
abilities to use and connect external representations.

On the other hand, representations play a central role in the Realistic Mathematics Education (RME). In RME, mathematics is viewed as an organizing activity that makes sense of the world, and learners are viewed as mathematicians who act as the reinventors of mathematics. By constructing, connecting, and evolving representations while being actively engaged in “realistic” contexts, learners construct or reconstruct mathematical ideas (Gravemeijer, 2008). As such, realistic tasks play a critical role in RME. Realistic tasks in RME are different from traditional word problems in that tasks serve as the start of knowledge construction instead of as the application of learned facts at the end of instruction. Realistic tasks have representations built-in, are truly problematic so that students have motivations to solve them, and are open enough so students can take various paths to solving the problems (Fosnot & Jacob, 2010; Gravemeijer, 2008).

In this study, I use the lenses of representation and realistic task—the lens of representation for teachers’ problem solving tendencies and abilities, and the lens of realistic task for task design and implementation—to examine teachers’ knowledge. As the participants in this study took various courses, designed following some principles of RME, where representations and realistic tasks were particularly emphasized, these lenses served as appropriate analytic tools to look into their knowledge.

I here begin with a brief literature review that regards secondary teachers’ tendencies and abilities to use representations in problem solving, as well as their conceptions about representations. The literature on teachers’ conceptions of or tendencies and abilities to use realistic tasks that is consistent with the RME description is rare at the secondary level, and I include two studies that show the effects of RME with Dutch secondary teachers.

**Teachers’ Conception, Use, and Understanding of Representations**

Studies have shown that many secondary teachers hold less than optimal conceptions or knowledge of representations in teaching and learning. They focus mostly on symbolic representations in teaching (Cunningham, 2005); they tend to use algebraic approaches in their own problem solving and favor students’ algebraic approaches over other approaches (Nathan & Petrosino, 2003); and they view non-symbolic representations as informal objects, not necessarily as mathematics itself (Stylianou, 2010). In Moyer’s study (2001), even after an intervention that emphasized the role of manipulatives and technology in instruction, many in-service middle grade teachers used manipulatives mainly in teacher-directed ways in their instruction and viewed the use of manipulatives as playing, exploring, or changing of pace, but not necessarily as making sense of mathematical concepts or ideas.

Many studies have also shown teachers’ lack of abilities to connect representations in problem solving. Hitt (1998) showed that many beginning secondary teachers were unable to articulate whether a curve represented by an algebraic form—circle or ellipse—was a function or not. Even (1998) showed that secondary teachers have difficulties in determining the number of zeros of an algebraic equation, \(y=ax^2+bx+c\), due to their inability to connect the equation to its graph. In Presmeg and Nenaduradu (2005), a teacher used mostly tabular/numerical representations in solving problems. Even for a problem in which he used a graphical representation, he came up with a line graph in spite of his table showing a quadratic pattern.

**Teachers’ Conceptions of Realistic Tasks**

In the Netherlands, the RME theory has been used in teacher education programs to prepare preservice teachers, and there have been some reports attesting to positive effects from the
programs. According to Korthagen and Kessels (1999), a national evaluation study of all Dutch education programs showed that preservice secondary teachers who were trained within the RME framework had higher ratings than their counterparts. Wobbles, Korthagen, and Broekman (1997) also showed that a group of preservice teachers trained in RME not only was much more in favor of RME principles than was a random group of experienced inservice teachers, but also characterized their teacher education program favorably—as using mathematics in contexts, having an inquiry approach, using mathematics as an activity, and using different explanations. However, when asked about important characteristics of “high school” mathematics education, only a few of the 10 preservice teachers mentioned that using mathematics in “realistic” contexts or mathematics as an activity are important characteristics.

As evidenced by research above, there is much to be done about secondary teachers’ conceptions of the roles of representations and realistic tasks and their abilities to use multiple representations in problem solving. Yet research documenting the kinds of interventions implemented and the effects of such interventions is very limited. This study concerns this under-documented area. The details of this study follow.

Methodology

Research Contexts

The subjects of this study are three preservice secondary mathematics teachers—Jennifer, Kristal, and Shane (pseudonyms)—who took four mathematics content and methods courses. As undergraduates, the teachers majored in mathematics with an emphasis in secondary teaching and minored in mathematics education. In their junior year, they participated in the mathematics content courses, Math A and B (pseudonyms). After completing their undergraduate degrees, they entered a teacher education program, offered at the same university, to obtain secondary teaching credentials in mathematics. During the teacher education program, they had two mathematics education courses, ED A and B (pseudonyms). The three participants were specifically chosen for this study because they were the only teachers in the secondary teaching cohort who had taken Math A and B and ED A and B.

The Math A and B courses had two main goals: helping future teachers reconceptualize the key concepts of grade 5-12 mathematics through the analysis of student work and tasks, and providing them with opportunities to construct mathematical ideas themselves through the problem solving of college level mathematics. Throughout the courses, the roles of representations and tasks in teaching and learning were emphasized through reading, discussion, problem solving, and task analysis by using Young Mathematicians at Work: Constructing Fractions Decimals and Percents (Fosnot & Dolk, 2002), Making Sense: Teaching and Learning Mathematics with Understanding (Hiebert et al., 1997), and the Contexts for Learning series. College level tasks based on historical contexts, such as Euclid, Archimedes, and Omar Khayyam as well as the tasks based on literature, such as Thompson (1994a) and Carlson et. al (2004), were also used so that preservice teachers themselves would experience construction of mathematical ideas. Most of the tasks used in the courses were realistic and naturally embedded representations. It is noteworthy, however, that much of the materials used in the courses were either at the grade 5-8 level or at the college level, due to their focus on student mathematics thinking at the grade 5-12 level and a lack of professional materials at the high school level that followed principles similar to RME.

The two graduate mathematics education courses, ED A and B (offered in the fall and winter quarters, respectively) focused mainly on grade 5-12 mathematics. The main activities for the
courses were problem solving and discussion of articles regarding many issues in mathematics teaching and learning. The two major components of the courses were *Young mathematicians at work*, *Constructing algebra* (Fosnot & Jacob, 2010) and the *Cookies* unit from a reform based high school curriculum, *Interactive Mathematics Program*. Much of the class hours were spent on solving algebra problems and discussing students’ construction of ideas in algebra. As a final portfolio, they submitted 15-20 pages of writing on the *Cookies* unit in which they explained how the unit brought out mathematical ideas—such as equivalence, inequality, variation, the Cartesian connection, and optimization—in relation to representations—such as the number line model and the Cartesian coordinate plane.

**Data and Analysis**

Two different types of data were used for analysis: two individual interviews and the Performance Assessment for California Teachers (PACT) teaching events. The first interview lasted about an hour and was conducted after the participants finished the Ed A course. The second interview lasted about 2 hours and was conducted while they were taking the Ed B course. Interview questions included items that could measure participants’ problem solving tendencies and abilities, their conceptions of representations, and their student teaching experience and PACT. Both interviews were recorded with a video camera and were transcribed. Participants’ written responses on the interviews were also collected. The PACT teaching event was collected at the end of their teacher education program.

There were 7 interview questions that regarded the first research question—“How were preservice secondary teachers’ tendencies and abilities to solve problems using representations after taking courses that emphasized the roles of representations and tasks?” The questions were similar or identical to items in the MKT survey (Hill et al., 2008) or tasks in Dufour-Janvier, Bednarz, and Belanger (1987), Even (1998), and Knuth (2000). For the grade 5-8 level, included were three problems related to a division of fractions, \( \frac{2}{3} \div \frac{1}{6} \); a multiplication of decimals, \( 1.5 \times 0.7 \); and an algebraic expression, \((x+2)(x+3)\). For the high school level, included were four problems related to the number of solutions of \( x^4 = x + 2 \); the number of solutions of \( \sqrt{x} = x - 2 \); the solutions of an equation, \( ?x + 3y = -2 \), with a graph of the equation provided; and the signs of \( a, b, \) and \( c \) in \( y = ax^2 + bx + c \), with a graph of the function provided. For analysis, I coded the kinds of approaches the participants used in problem solving and the correctness of their work.

For the second research question—“How were preservice secondary teachers’ abilities to use tasks in instruction after taking courses that emphasized the roles of representations and tasks?”, PACT events as well as interview items that were relevant to task design and implementation were analyzed. Two interview items asked the participants to generate a task and to explain how they would implement the task for students who had little instruction on the mathematical topics of linear functions and of quadratic functions. Three interview items asked them to explain the roles of representations, the roles of realistic tasks, and their experiences in student teaching. The PACT event was a collection of instructional materials—video clips of teaching segments, students’ work samples, daily reflections on instruction, and commentaries—from a week-long learning segment of instruction during their student teaching. As such, it showed participants’ abilities to design and implement tasks in instruction. Among multiple tasks included in the PACT event, I selected major tasks for which the preservice teachers spent 30 minutes or longer of instructional time. Because an implementation of realistic tasks normally requires a considerable amount of time, those tasks had a better chance to show participants’ abilities to use
representations and realistic tasks. Jessica and Kristal had two tasks and Shane had one task that met the selection criteria.

For the analysis of data related to the second research question, I modified the Context Scale of the Assessment of Facilitation of Mathematizing (AFM; Fosnot, Dolk, Zolkower, & Seignoret, 2006). The AFM Context Scale was a measuring tool, examining teachers’ abilities to use contexts in teaching on three levels based on teachers’ use of representations and tasks. Rather than classifying the tasks into three levels as in the AFM Scale, I classified the participants’ tasks into two levels. The discriminating factor between the two levels was how they used tasks in instruction: If a task was used as an application of learned facts, it was rated Level 1; if a task was used as a departure of students’ construction of mathematical ideas, it was rated Level 2.

Results

Tendencies and Abilities to Use Representations in Problem Solving

Analysis showed that despite taking the courses that focused on multiple representations and the connections among representations, preservice teachers showed tendencies to use traditional, algebraic approaches in problem solving for both grade 5-8 and high school level problems when they were not prompted to use visual/geometric representations. In addition, when they were prompted to use visual/geometric representations, they were more successful on the grade 5-8 level problems than on the high school level problems.

As for their tendencies to use visual/graphical representations, for each of the three grade 5-8 level problems, only one participant used a visual representation to answer the question when not prompted to do so. Kristal used an area model to explain why \((x+2)(x+3)\) is equivalent to \(x^2+5x+6\), and Jennifer and Shane used the FOIL method to explain it. The same trait was shown for the questions regarding the division of fractions, \(2/3 ÷ 1/6\), and the multiplication of decimals, \(1.5 \times .7\). Shane used number line models to explain both operations, but Jennifer and Kristal used the invert and multiply method for the fraction problem and the multiplication algorithm for the decimal problem. Their tendencies to use algebraic approaches on the three high school level algebra problems were higher than those on the grade 5-8 level problems. Except for the case where Shane used a graph to answer one of the three problems, all three participants used algebraic approaches to answer the high school level questions. All three used algebraic approaches and successfully found a solution of \(?x + 3y = -2\); Jessica and Shane used algebraic approaches and successfully determined the number of solutions of \(\sqrt{x} = x - 2\); Jessica and Kristal used algebraic approaches, but failed to determine the number of solutions of the equation, \(x^4=-x+2\).

Despite having tendencies to use traditional, algebraic approaches for the grade 5-8 level problems, when they were prompted to use visual/geometric representations, they came up with representations to explain the grade 5-8 level problems. All three participants used area models to explain \(1.5 \times .7\) and \((x+2)(x+3) = x^2+5x+6\), and Kristal and Shane used a number line model and an area model to explain \(2/3 ÷ 1/6\). However, they were less successful in solving high school level problems using geometric representations, even with prompts. Although all three participants provided a solution successfully by examining the graph of \(?x + 3y = -2\), when prompted, they could not determine the number of solutions of \(\sqrt{x} = x - 2\) or of \(x^4=-x+2\) by using graphs. A major obstacle for them was that they did not know where to start. Kristal and Jessica said that with no y present in the equations, they did not know what graphs had to be
considered. Yet when they were told that graphs of \( y = x^4 \) and \( y = -x + 2 \) could be considered, they sketched the graphs and explained using graphs that there were two real solutions of \( x^4 = -x + 2 \). In the question where they had to determine the signs of \( a \), \( b \), and \( c \) in \( y = ax^2 + bx + c \), using the given graph, none of them could determine the sign of \( b \) successfully. They could determine the sign of \( a \) from the concave down shape of the parabola graph, but they were unable to use the location of the vertex to determine the sign of \( b \).

It was noteworthy that they were unable to associate graphs to the equations, \( x^4 = -x + 2 \) and \( \sqrt{x} = x - 2 \), after taking many courses that emphasized the connections among representations. It was also noteworthy that their positive conceptions of representations did not help them to solve the problems. They said in an interview: “Graphs are important since they are used in all sorts of things. But if you don’t understand what graphs mean, then the graph has no meanings and it can’t serve its purpose”; “a teacher’s knowing many ways of representing is important because students learn differently”; and “representing in many ways is important because there would be some students who might not understand in one representation.” However when they were given tasks in which they had to solve by connecting representations, they were unable to do so.

**Knowledge of Tasks**

Their abilities to design and use realistic tasks shown in the two interviews were quite satisfactory, with all of them rated Level 2. For the linear function topic, Jennifer used a money context that showed a constant rate of change of 5; Kristal used a paper-cutting context which led to the linear function \( y = 2x \); and Shane used a context involving two different constant rates of change of jumps. All of them also explained that they were using the contexts as the start of the construction of the ideas such as slope, rate of change, and the initial value, by connecting tables and graphs to the problem contexts. For the quadratic function topic, Jennifer used a square number sequence context of 1, 4, 9, … and emphasized that the differences in the steps formed a linear function; Kristal used \( y = x^2 \) by connecting it to the linear function \( y = x \) and \( y = x^2 + c \) graphs; and Shane used a tile context in which the \( n \)th step included \( n^2 \) number of tiles. All of them explained that they were using the tasks as a departure of a meaning-making process, focusing on the development of concepts such as the rate of change of a quadratic function as a linear function or its connection to linear function by connecting tables, graphs, and problem contexts.

The tasks in their PACT events were rated lower than those on the linear and quadratic function concepts. Jennifer had two word problem tasks on linear functions, rated Level 2, which were similar to the tasks she provided in an interview. Kristal had two hands-on activities on linear inequality, rated Level 1, which she used as a practice of learned rules. Shane had a hands-on task on parallelogram, rated Level 1, which was basically a parallelogram constructing activity using the conditions of parallelogram learned in previous classes. Jennifer was the only one who used tasks as the departure of meaning-making process, and Kristal and Shane used the tasks as applications of learned facts.

Some factors that contributed to the lower ratings on PACT tasks were their students’ academic levels (or their beliefs about how students with lower academic abilities should be taught) and their lack of understanding about realistic tasks. In Kristal’s case, for example, although she believed that students in general needed to be given opportunities to construct ideas themselves, she learned from her teaching that her low academic level students could not learn mathematics in that way. She thus adjusted her teaching style, following behaviorism principles, and used mostly skill-based or low-level tasks in her instruction. Another contributing factor was
their lack of understanding of “realistic” tasks. All three described realistic tasks as “real-life,” “hands on,” “fun,” “interesting,” “something that made them engaged,” “making them want to learn math,” and “incorporate something hands-on.” But none of them mentioned that the use of tasks as the starting point of constructing mathematical ideas was important in task implementation. As such, when they were asked in interviews to design tasks on the topics with which they were familiar through the intervention courses, such as linear function and quadratic function, they came up with tasks that were similar to the tasks used in the courses and explained in the ways that they would use the tasks as the departure of understanding. But when they were given a topic with which they were unfamiliar, such as parallelogram in Shane’s case, they focused on the fun, hands-on, interesting, and real-life part of tasks rather than on the use of tasks as the departure of construction of ideas.

**Discussion and Conclusions**

This study examined three preservice secondary teachers’ problem solving tendencies and abilities to use representations as well as their abilities to use tasks in instruction after taking a series of mathematics and mathematics education courses that focused on representations and realistic tasks. The results of this study are bi-fold. As for their abilities and tendencies to use representations, on the positive side, they were able to solve problems or to explain ideas using representations other than symbolic for the grade 5-8 level problems when they were prompted to do. On the negative side, however, they could not solve high school level problems by connecting algebraic and geometric representations as in the teachers in Gagatsis and Shiakalli (2004) and Presmeg and Nenduradu (2005), even when they were prompted to do so. Algebra problems requiring the connections between symbolic and graphical representations, such as those in Dufour-Janvier et al. (1987) and Even (1998), were very challenging to them. Furthermore, when they were not prompted to use visual/geometric representations or approaches, they still showed tendencies to use algebraic or traditional approaches as in teachers in Nathan and Petrosino (2003).

As for their use of tasks in instruction, on the positive side, they developed beliefs that realistic tasks or multiple representations are important parts of learning. Unlike the grade 5-8 teachers in Moyer (2002) and Stylianout (2010) or in Wobbles, Korthagen, and Broekman (1997), the preservice secondary teachers in this study believed that it was important to understand mathematical ideas in many representational contexts, that students’ construction of ideas was an important characteristic of secondary mathematics, and that realistic tasks were important. They also designed many tasks that embedded multiple representations that could potentially help students construct critical ideas. Yet on the negative side, they understood realistic tasks as tasks that were related to real-life, fun, and interesting, but not as tasks that were used as the departure of construction of ideas.

This study suggests that preparing secondary teachers should be done with special care. Teachers need to be provided with opportunities to design or modify tasks using traditional mathematics curriculum or ill-designed tasks. As shown in this study, it is difficult for them to design and implement tasks that can lead to students’ construction of ideas on unfamiliar topics, if they learn the importance of tasks only with well-designed tasks. By having opportunities to discuss deficits in ill-designed tasks and to modify the ill-designed tasks into well-designed realistic tasks, they might develop knowledge of tasks that can be transferred into their teaching in various situations.
References


